

# ROSENBLATT DISTRIBUTION SUBORDINATED TO GAUSSIAN RANDOM FIELDS WITH LONG-RANGE DEPENDENCE

N.N. LEONENKO, M.D. RUIZ-MEDINA, AND M.S. TAQQU

**ABSTRACT.** The Karhunen-Loève expansion and the Fredholm determinant formula are used, to derive an asymptotic Rosenblatt-type distribution of a sequence of integrals of quadratic functions of Gaussian stationary random fields on  $\mathbb{R}^d$  displaying long-range dependence. This distribution reduces to the usual Rosenblatt distribution when  $d = 1$ . Several properties of this new distribution are obtained. Specifically, its series representation, in terms of independent chi-squared random variables, is established. Its Lévy-Khintchine representation, and membership to the Thorin subclass of self-decomposable distributions are obtained as well. The existence and boundedness of its probability density then follow as a direct consequence.

*Keywords.* Fredholm determinant; Hermite polynomials; infinite divisible distributions; multiple Wiener-Itô stochastic integrals; non-central limit theorems; Rosenblatt-type distribution.

*AMS subject classifications.* 60F99; 60E10; 60G15; 60G60.

## 1. INTRODUCTION

The aim of this paper is to derive and study the properties of the limit distribution, as  $T \rightarrow \infty$ , of the random integral

$$(1.1) \quad S_T = \frac{1}{d_T} \int_{D(T)} (Y^2(\mathbf{x}) - 1) d\mathbf{x},$$

where the normalizing function  $d_T$  is given by

$$(1.2) \quad d_T = T^{d-\alpha} \mathcal{L}(T), \quad 0 < \alpha < d/2,$$

with  $\mathcal{L}$  being a positive slowly varying function at infinity, that is

$$(1.3) \quad \lim_{T \rightarrow \infty} \mathcal{L}(T\|\mathbf{x}\|)/\mathcal{L}(T) = 1,$$

for every  $\|\mathbf{x}\| > 0$ , and  $D(T) \subset \mathbb{R}^d$  denotes a homothetic transformation of a set  $D \subset \mathbb{R}^d$ , with center at the point  $\mathbf{0} \in D$ , and coefficient or scale factor  $T > 0$ . In the subsequent development,  $D$  is assumed to be a regular bounded domain, whose interior has positive Lebesgue measure, and with boundary having null Lebesgue measure. Here,  $\{Y(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$  is a zero-mean Gaussian homogeneous and isotropic random field with values in  $\mathbb{R}$ , displaying long-range dependence. That is,  $Y$  is assumed to satisfy the following condition:

**Condition A1.** The random field  $\{Y(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$  is a measurable zero-mean Gaussian homogeneous and isotropic mean-square continuous random field on a probability space  $(\Omega, \mathcal{A}, P)$ ,

---

*Date:* 27 May 2016.

This work has been supported in part by project MTM2015-71839-P (co-funded by European Regional Development Funds), MINECO, Spain. This research was also supported under Australian Research Council's Discovery Projects funding scheme (project number DP160101366), and under Cardiff Incoming Visiting Fellowship Scheme and International Collaboration Seedcorn Fund. Murad S. Taqqu was supported in part by the NSF grants DMS-1007616 and DMS-1309009 at Boston University.

with  $EY^2(\mathbf{x}) = 1$ , for all  $\mathbf{x} \in \mathbb{R}^d$ , and correlation function  $E[Y(\mathbf{x})Y(\mathbf{y})] = B(\|\mathbf{x} - \mathbf{y}\|)$  of the form:

$$(1.4) \quad B(\|\mathbf{z}\|) = \frac{\mathcal{L}(\|\mathbf{z}\|)}{\|\mathbf{z}\|^\alpha}, \quad \mathbf{z} \in \mathbb{R}^d, \quad 0 < \alpha < d/2.$$

From **Condition A1**, the correlation  $B$  of  $Y$  is a continuous function of  $r = \|\mathbf{z}\|$ . It then follows that  $\mathcal{L}(r) = \mathcal{O}(r^\alpha)$ ,  $r \rightarrow 0$ . Note that the covariance function

$$(1.5) \quad B(\|\mathbf{z}\|) = \frac{1}{(1 + \|\mathbf{z}\|^\beta)^\gamma}, \quad 0 < \beta \leq 2, \quad \gamma > 0,$$

is a particular case of the family of covariance functions (1.4) studied here with  $\alpha = \beta\gamma$ , and

$$(1.6) \quad \mathcal{L}(\|\mathbf{z}\|) = \|\mathbf{z}\|^{\beta\gamma} / (1 + \|\mathbf{z}\|^\beta)^\gamma.$$

The limit random variable of (1.1) will be denoted as  $S_\infty$ . The distribution of  $S_\infty$  will be referred to as the *Rosenblatt-type* distribution, or sometimes simply as the *Rosenblatt* distribution because this is how it is known in the case  $d = 1$ . In that case, a discretized version in time of the integral (1.1) first appears in [35], and the limit functional version is considered in [42] in the form of the Rosenblatt process. In this classical setting, the limit of (1.1) is represented by a double Wiener-Itô stochastic integral (see [12]; [43]). Other relevant references include, for example, [2], [3], [15], [17], [24], [36], to mention just a few. The general approach considered here for deriving the weak-convergence to the Rosenblatt distribution is inspired by [42], which is based on the convergence of characteristic functions. This approach has also been used, recently, in [24], to study the characteristic functions of quadratic forms of strongly-correlated Gaussian random variables sequences.

We suppose here  $d \geq 2$ , and thus consider integrals of quadratic functions of long-range dependence zero-mean Gaussian stationary random fields. We pursue, however, a different methodology than in the case  $d = 1$ , which was based on the discretization of the parameter space. A direct extension of these techniques is not available when  $d \geq 2$ . Instead of discretizing the parameter space of the random field, we focus on the characteristic function for quadratic forms for Hilbert-valued Gaussian random variables (see, for example, [11]), and take advantage of functional analytical tools, like the Karhunen-Loève expansion and the Fredholm determinant formula, to obtain the convergence in distribution to a limit random variable  $S_\infty$  with Rosenblatt-type distribution.

The double Wiener-Itô stochastic integral representation of  $S_\infty$  in the spectral domain leads to its series expansion in terms of independent chi-squared random variables, weighted by the eigenvalues of the integral operator introduced in equation (3.1) below. The asymptotics of these eigenvalues is given in Corollary 2. The infinitely divisible property of  $S_\infty$  is then obtained as a direct consequence of the previous results derived, in relation to the series expansion of  $S_\infty$ , and the asymptotic properties of the eigenvalues. We also prove that the distribution of  $S_\infty$  is self-decomposable, and that it belongs, in particular, to the Thorin subclass. The existence and boundedness of the probability density of  $S_\infty$  then follows.

The outline of the paper is now described. In Section 2, we recall the Karhunen-Loève expansion, introduce the Fredholm determinant formula, and use the referred tools to obtain the characteristic function of (1.1). In Section 3, we prove the weak convergence of (1.1) to the random variable  $S_\infty$  with a Rosenblatt-type distribution. The double Wiener-Itô stochastic integral representation of  $S_\infty$ , its series expansion in terms of independent chi-square random variables, and the asymptotics of the involved eigenvalues are established in Section 4. These results are applied in Section 5 to derive some properties of the Rosenblatt distribution, e.g., infinitely divisible property, self-decomposability, and, in particular, the membership to the Thorin subclass. Appendices A-C provide some auxiliary results and the proofs of some propositions and corollaries.

In this paper we consider the case of real-valued random fields. In what follows we use the symbols  $C, C_0, M_1, M_2$ , etc., to denote constants. The same symbol may be used for different constants appearing in the text.

## 2. KARHUNEN-LOÉVE EXPANSION AND RELATED RESULTS

This section introduces some preliminary definitions, assumptions and lemmas hereafter used in the derivation of the main results of this paper. We start with the Karhunen-Loève Theorem for a zero-mean second-order random field  $\{Y(\mathbf{x}), \mathbf{x} \in K \subset \mathbb{R}^d\}$ , with continuous covariance function  $B_0(\mathbf{x}, \mathbf{y}) = E[Y(\mathbf{x})Y(\mathbf{y})]$ ,  $(\mathbf{x}, \mathbf{y}) \in K \times K \subset \mathbb{R}^d \times \mathbb{R}^d$ , defined on a compact set  $K$  of  $\mathbb{R}^d$  (see Section 3.2 in [1]). This theorem provides the following orthogonal expansion of the random field  $Y$  :

$$(2.1) \quad \begin{aligned} Y(\mathbf{x}) &= \sum_{j=1}^{\infty} \sqrt{\lambda_j} \phi_j(\mathbf{x}) \eta_j, \quad \mathbf{x} \in K, \\ \lambda_k \phi_k(\mathbf{x}) &= \int_K B_0(\mathbf{x}, \mathbf{y}) \phi_k(\mathbf{y}) d\mathbf{y}, \quad k \in \mathbb{N}_*, \quad \langle \phi_i, \phi_j \rangle_{L^2(K)} = \delta_{i,j}, \quad i, j \in \mathbb{N}_*, \end{aligned}$$

where  $\eta_k = \frac{1}{\sqrt{\lambda_k}} \int_K Y(\mathbf{x}) \phi_k(\mathbf{x}) d\mathbf{x}$ , for each  $k \geq 1$ , and the convergence holds in the  $L^2(\Omega, \mathcal{A}, P)$  sense. The eigenvalues of  $B_0$  are considered to be arranged in decreasing order of magnitude, that is,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{k-1} \geq \lambda_k \geq \dots$ . The orthonormality of the eigenfunctions  $\phi_j$ ,  $j \in \mathbb{N}_*$ , leads to the uncorrelation of the random variables  $\eta_j$ ,  $j \in \mathbb{N}_*$ , with variance one, since

$$E[\eta_j \eta_k] = \int_K \int_K B_0(\mathbf{x}, \mathbf{y}) \phi_j(\mathbf{y}) \phi_k(\mathbf{x}) d\mathbf{y} d\mathbf{x} = \lambda_j \int_K \phi_j(\mathbf{x}) \phi_k(\mathbf{x}) d\mathbf{x} = \lambda_j \delta_{j,k},$$

with  $\delta$  denoting the Kronecker delta function. In the Gaussian case, they are independent.

For each  $T > 0$ , let us fix some notation related to the Karhunen-Loève expansion of the restriction to the set  $D(T)$  of Gaussian random field  $Y$ , with covariance function (1.4). By  $R_{Y,D(T)}$  we denote the covariance operator of  $Y$  with covariance kernel  $B_{0,T}(\mathbf{x}, \mathbf{y}) = E[Y(\mathbf{x})Y(\mathbf{y})]$ ,  $\mathbf{x}, \mathbf{y} \in D(T)$ , which, as an operator from  $L^2(D(T))$  onto  $L^2(D(T))$ , satisfies

$$R_{Y,D(T)}(\phi_{l,T})(\mathbf{x}) = \int_{D(T)} B_{0,T}(\mathbf{x}, \mathbf{y}) \phi_{l,T}(\mathbf{y}) d\mathbf{y} = \lambda_{l,T}(R_{Y,D(T)}) \phi_{l,T}(\mathbf{x}), \quad l \in \mathbb{N}_*,$$

where, in the following, by  $\lambda_k(A)$  we will denote the  $k$ th eigenvalue of the operator  $A$ . In particular,  $\{\lambda_{k,T}(R_{Y,D(T)})\}_{k=1}^{\infty}$  and  $\{\phi_{k,T}\}_{k=1}^{\infty}$  respectively denote the eigenvalues and eigenfunctions of  $R_{Y,D(T)}$ , for each  $T > 0$ . Note that, as commented,  $B_{0,T}$  refers to the covariance function of  $\{Y(\mathbf{x}), \mathbf{x} \in D(T)\}$  as a function of  $(\mathbf{x}, \mathbf{y}) \in D(T) \times D(T)$ , which, under **Condition A1**, defines a non-negative, symmetric and continuous kernel on  $D(T)$ , satisfying the conditions assumed in Mercer's Theorem. Hence, the Karhunen-Loève expansion of random field  $Y$  holds on  $D(T)$ , and its covariance kernel  $B_{0,T}$  also admits the series representation

$$(2.2) \quad B_{0,T}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)}) \phi_{j,T}(\mathbf{x}) \phi_{j,T}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in D(T),$$

where the convergence is absolute and uniform (see, for example, [1], pp.70-74). The orthonormality of the eigenfunctions  $\{\phi_{l,T}\}_{l=1}^{\infty}$  yields

$$(2.3) \quad \frac{1}{d_T} \int_{D(T)} Y^2(\mathbf{x}) d\mathbf{x} = \frac{1}{d_T} \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)}) \eta_{j,T}^2.$$

In the derivation of the limit characteristic function of (1.1), we will use the Fredholm determinant formula of a trace operator. Recall first that a positive operator  $A$  on a separable Hilbert space  $H$  is a trace operator if

$$(2.4) \quad \|A\|_1 \equiv \text{Tr}(A) \equiv \sum_k \left\langle (A^*A)^{1/2} \varphi_k, \varphi_k \right\rangle_H < \infty,$$

where  $A^*$  denotes the adjoint of  $A$  and  $\{\varphi_k\}$  is an orthonormal basis of the Hilbert space  $H$  (see [34], pp. 207-209). A sufficient condition for a compact and self-adjoint operator  $A$  to belong to the trace class is  $\sum_{k=1}^{\infty} \lambda_k(A) < \infty$ . For each finite  $T > 0$ , the operator  $R_{Y,D(T)}$  is in the trace class, since from equation (2.2), applying the orthonormality of the eigenfunction system  $\{\phi_{j,T}, j \in \mathbb{N}_*\}$ , and keeping in mind that  $B_{0,T}(\mathbf{0}) = 1$ , we have

$$(2.5) \quad \text{Tr}(R_{Y,D(T)}) = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)}) = \int_{D(T)} B_{0,T}(\mathbf{x}, \mathbf{x}) d\mathbf{x} = \int_{D(T)} d\mathbf{x} = T^d |D| < \infty,$$

where  $|D|$  denotes the Lebesgue measure of the compact set  $D$ . Note that the class of compact and self-adjoint operators contains the class of trace and self-adjoint operators. Hence, under **Condition A1**, from equation (2.5), the restriction of  $Y$  to  $D(T)$  admits a Karhunen-Lo  ve expansion, convergent in the mean-square sense (i.e., in the  $L^2(\Omega, \mathcal{A}, P)$ -sense), for any  $T > 0$ , and for an arbitrary regular bounded domain  $D$ . Furthermore, for any  $k \geq 1$ ,

$$(2.6) \quad R_{Y,D(T)}^k f(\mathbf{x}) = \int_{D(T)} B_{0,T}^{*(k)}(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \quad f \in L^2(D(T)),$$

where  $B_{0,T}^{*(k)}$  denotes

$$(2.7) \quad \begin{aligned} B_{0,T}^{*(1)}(\mathbf{x}, \mathbf{y}) &= B_{0,T}(\mathbf{x}, \mathbf{y}), \quad k = 1, \\ B_{0,T}^{*(k)}(\mathbf{x}, \mathbf{y}) &= \int_{D(T)} B_{0,T}^{*(k-1)}(\mathbf{x}, \mathbf{z}) B_{0,T}(\mathbf{z}, \mathbf{y}) d\mathbf{z}, \quad k = 2, 3, \dots \end{aligned}$$

From equations (2.2) and (2.7), applying the orthonormality of  $\phi_{j,T}$ ,  $j \in \mathbb{N}_*$ , one can obtain

$$(2.8) \quad \text{Tr}(R_{Y,D(T)}^k) = \sum_{j=1}^{\infty} \lambda_{j,T}^k(R_{Y,D(T)}) = \int_{D(T)} B_{0,T}^{*(k)}(\mathbf{x}, \mathbf{x}) d\mathbf{x} < \infty, \quad k \in \mathbb{N}_*,$$

since, for every  $k \geq 1$ ,  $|\lambda_k(R_{Y,D(T)})| \leq M |\lambda_k(R_{Y,D(T)})|^k = M |\lambda_k(R_{Y,D(T)}^k)|$ , for some positive constant  $M$ . In particular, in the homogeneous random field case,

$$(2.9) \quad \begin{aligned} \text{Tr}(R_{Y,D(T)}^k) &= \sum_{j=1}^{\infty} \lambda_{j,T}^k(R_{Y,D(T)}) = \int_{D(T)} B_{0,T}^{*(k)}(\mathbf{x}_k, \mathbf{x}_k) d\mathbf{x}_k \\ &= \int_{D(T)} \dots \int_{D(T)} \left[ \prod_{j=1}^{k-1} B_{0,T}(\mathbf{x}_{j+1} - \mathbf{x}_j) \right] B_{0,T}(\mathbf{x}_1 - \mathbf{x}_k) d\mathbf{x}_1 \dots d\mathbf{x}_k, \end{aligned}$$

and, in the homogeneous and isotropic case, for  $k = 2$ ,

$$(2.10) \quad \text{Tr}(R_{Y,D(T)}^2) = \sum_{j=1}^{\infty} \lambda_{j,T}^2(R_{Y,D(T)}) = \int_{D(T)} \int_{D(T)} \frac{\mathcal{L}^2(\|\mathbf{x} - \mathbf{y}\|)}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x}.$$

The following definition introduces the Fredholm determinant of an operator  $A$ , as a complex-valued function which generalizes the determinant of a matrix.

**Definition 1.** (see, for example, [39], Chapter 5, pp.47-48, equation (5.12)) Let  $A$  be a trace operator on a separable Hilbert space  $H$ . The Fredholm determinant of  $A$  is

$$(2.11) \quad \mathcal{D}(\omega) = \det(I - \omega A) = \exp \left( - \sum_{k=1}^{\infty} \frac{\text{Tr} A^k}{k} \omega^k \right) = \exp \left( - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [\lambda_l(A)]^k \frac{\omega^k}{k} \right),$$

for  $\omega \in \mathbb{C}$ , and  $|\omega| \|A\|_1 < 1$ . Note that  $\|A^m\|_1 \leq \|A\|_1^m$ , for  $A$  being a trace operator.

**Lemma 1.** Let  $\{Y(\mathbf{x}), \mathbf{x} \in D \subset \mathbb{R}^d\}$  be an integrable and continuous, in the mean-square sense, zero-mean, Gaussian random field, on a bounded regular domain  $D \subseteq \mathbb{R}^d$  containing the point zero. Then, the following identity holds:

$$(2.12) \quad \begin{aligned} E \left[ \exp \left( i\xi \int_D Y^2(\mathbf{x}) d\mathbf{x} \right) \right] &= \prod_{j=1}^{\infty} (1 - 2\lambda_j(R_{Y,D}) i\xi)^{-1/2} = (\mathcal{D}(2i\xi))^{-1/2} \\ &= \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \frac{(2i\xi)^m}{m} \text{Tr}(R_{Y,D}^m) \right), \end{aligned}$$

for  $\|R_{Y,D}\|_1 |2i\xi| < 1$ , as given in Definition 1.

*Proof.* The covariance operator  $R_{Y,D}$  of  $Y$ , acting on the space  $L^2(D)$ , is in the trace class. From Definition 1, the following identities hold:

$$(2.13) \quad \begin{aligned} E \left[ \exp \left( i\xi \int_D Y^2(\mathbf{x}) d\mathbf{x} \right) \right] &= E \left[ \exp \left( i\xi \sum_{j=1}^{\infty} \lambda_j(R_{Y,D}) \eta_j^2 \right) \right] \\ &= \prod_{j=1}^{\infty} E \left[ \exp \left( i\xi \lambda_j(R_{Y,D}) \eta_j^2 \right) \right] = \prod_{j=1}^{\infty} (1 - 2\lambda_j(R_{Y,D}) i\xi)^{-1/2} = (\mathcal{D}(2i\xi))^{-1/2} \\ &= \left[ \exp \left( - \sum_{m=1}^{\infty} \frac{(2i\xi)^m}{m} \text{Tr}(R_{Y,D}^m) \right) \right]^{-1/2} = \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \frac{(2i\xi)^m}{m} \text{Tr}(R_{Y,D}^m) \right), \end{aligned}$$

where the last two identities in equation (2.13) are finite for  $|\xi| < \frac{1}{2\|D\|}$ , from the Fredholm determinant formula (2.11). Note that

$$(2.14) \quad \text{Tr}(R_{Y,D}^m) = \sum_{j=1}^{\infty} \lambda_j^m(R_{Y,D}) \leq \lambda_1^{m-1}(R_{Y,D}) \sum_{j=1}^{\infty} \lambda_j(R_{Y,D}) = \lambda_1^{m-1}(R_{Y,D}) \|R_{Y,D}\|_1 < \infty.$$

□

**Remark 1.** Similarly to equation (2.12), one can obtain the following identities, which will be used in the subsequent development: For a homothetic transformation  $D(T)$  of  $D \subset \mathbb{R}^d$ , with center at the point  $\mathbf{0} \in D$ , and coefficient  $T > 0$ ,

$$(2.15) \quad \begin{aligned} E \left[ \exp \left( i\xi \int_{D(T)} Y^2(\mathbf{x}) d\mathbf{x} \right) \right] &= \prod_{j=1}^{\infty} (1 - 2\lambda_{j,T}(R_{Y,D(T)}) i\xi)^{-1/2} = (\mathcal{D}_T(2i\xi))^{-1/2} \\ &= \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \frac{(2i\xi)^m}{m} \text{Tr}(R_{Y,D(T)}^m) \right), \end{aligned}$$

where  $\lambda_{1,T}(R_{Y,D(T)}) \geq \lambda_{2,T}(R_{Y,D(T)}) \geq \dots \geq \lambda_{j,T}(R_{Y,D(T)}) \geq \dots$ , with, as before,  $\{\lambda_{j,T}(R_{Y,D(T)}), j \in \mathbb{N}_*\}$  denoting the system of eigenvalues of the covariance operator  $R_{Y,D(T)}$  of  $Y$ , as an operator from  $L^2(D(T))$  onto  $L^2(D(T))$ . The last identity in equation (2.15) holds for  $\|R_{Y,D(T)}\|_1 |2i\xi| < 1$ , i.e., for  $\text{Tr}(R_{Y,D(T)}) |2i\xi| = T^d \|D\| |2i\xi| < 1$ , or equivalently for  $|\xi| < \frac{1}{2T^d \|D\|}$ .

### 3. WEAK CONVERGENCE OF THE RANDOM INTEGRAL $S_T$

This section provides the weak convergence of the random integral (1.1) to a Rosenblatt-type distribution, in Theorem 2. This results is based on the asymptotic behavior of the eigenvalues of the integral operator  $\mathcal{K}_\alpha$  (see Theorem 1 below)

$$(3.1) \quad \mathcal{K}_\alpha(f)(\mathbf{x}) = \int_D \frac{1}{\|\mathbf{x} - \mathbf{y}\|^\alpha} f(\mathbf{y}) d\mathbf{y}, \quad \forall f \in \text{Supp}(\mathcal{K}_\alpha), \quad 0 < \alpha < d,$$

with  $\text{Supp}(A)$  denoting the support of operator  $A$ . Operator (3.1) can be related with the Riesz potential  $(-\Delta)^{-\beta/2}$  of order  $\beta$ ,  $0 < \beta < d$ , on  $\mathbb{R}^d$ , formally defined as (see [41], p.117)

$$(3.2) \quad (-\Delta)^{-\beta/2}(f)(\mathbf{x}) = \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^d} \|\mathbf{x} - \mathbf{y}\|^{-d+\beta} f(\mathbf{y}) d\mathbf{y},$$

where  $(-\Delta)$  denotes the negative Laplacian operator, and

$$(3.3) \quad \gamma(\beta) = \frac{\pi^{d/2} 2^\beta \Gamma(\beta/2)}{\Gamma\left(\frac{d-\beta}{2}\right)} = \frac{1}{c(d, \beta)}, \quad 0 < \beta < d.$$

Indeed, except a constant, the function  $(1/\|\mathbf{x} - \mathbf{y}\|^\alpha)$  in equation (3.1) defines the kernel of the Riesz potential  $(-\Delta)^{(\alpha-d)/2}$  of order  $\beta = (d - \alpha)$ , for  $0 < \alpha < d$ . Similarly,  $(1/\|\mathbf{x} - \mathbf{y}\|^{2\alpha})$  is the kernel of the Riesz potential  $(-\Delta)^{\alpha-d/2}$  of order  $\beta = (d - 2\alpha)$  on  $\mathbb{R}^d$ , for  $0 < \alpha < d/2$ .

Recall that the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  is the space of infinitely differentiable functions on  $\mathbb{R}^d$ , whose derivatives remain bounded when multiplied by polynomials, i.e., whose derivatives are rapidly decreasing. Particularly,  $C_0^\infty(D) \subset \mathcal{S}(\mathbb{R}^d)$ , with  $C_0^\infty(D)$  denoting the infinitely differentiable functions with compact support contained in  $D$ .

The Fourier transform of the Riesz potential is understood in the weak sense, considering the space  $\mathcal{S}(\mathbb{R}^d)$ . The following lemma provides such a transform (see Lemma 1 of [41], p.117):

**Lemma 2.** *Let us consider  $0 < \beta < d$ .*

(i) *The Fourier transform of the function  $\|\mathbf{z}\|^{-d+\beta}$  is  $\gamma(\beta)\|\mathbf{z}\|^{-\beta}$ , in the sense that*

$$(3.4) \quad \int_{\mathbb{R}^d} \|\mathbf{z}\|^{-d+\beta} \overline{\psi(\mathbf{z})} d\mathbf{z} = \int_{\mathbb{R}^d} \gamma(\beta) \|\mathbf{z}\|^{-\beta} \overline{\mathcal{F}(\psi)(\mathbf{z})} d\mathbf{z}, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d),$$

where

$$\mathcal{F}(\psi)(\mathbf{z}) = \int_{\mathbb{R}^d} \exp(-i\langle \mathbf{x}, \mathbf{z} \rangle) \psi(\mathbf{x}) d\mathbf{x}$$

denotes the Fourier transform of  $\psi$ .

(ii) *The identity  $\mathcal{F}((-\Delta)^{-\beta/2}(f))(\mathbf{z}) = \|\mathbf{z}\|^{-\beta} \mathcal{F}(f)(\mathbf{z})$  holds in the sense that*

$$(3.5) \quad \int_{\mathbb{R}^d} (-\Delta)^{-\beta/2}(f)(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x}) \|\mathbf{x}\|^{-\beta} \overline{\mathcal{F}(g)(\mathbf{x})} d\mathbf{x}, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

In particular, the following convolution formula is obtained by iteration of (3.5) using (3.2):

$$\begin{aligned} & \int_{\mathbb{R}^d} \left( \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^d} \|\mathbf{x} - \mathbf{y}\|^{-d+\beta} \left[ \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^d} \|\mathbf{y} - \mathbf{z}\|^{-d+\beta} f(\mathbf{z}) d\mathbf{z} \right] d\mathbf{y} \right) \overline{g(\mathbf{x})} d\mathbf{x} \\ &= \int_{\mathbb{R}^d} (-\Delta)^{-\beta/2} \left[ (-\Delta)^{-\beta/2}(f) \right] (\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left[ \mathcal{F}((-\Delta)^{-\beta/2}(f))(\mathbf{x}) \right] \|\mathbf{x}\|^{-\beta} \overline{\mathcal{F}(g)(\mathbf{x})} d\mathbf{x} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x}) \|\mathbf{x}\|^{-\beta} \|\mathbf{x}\|^{-\beta} \overline{\mathcal{F}(g)(\mathbf{x})} d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x}) \|\mathbf{x}\|^{-2\beta} \overline{\mathcal{F}(g)(\mathbf{x})} d\mathbf{x} \\
&= \int_{\mathbb{R}^d} (-\Delta)^{-\beta}(f)(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x}, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \quad 0 < \beta < d/2,
\end{aligned}
\tag{3.6}$$

where we have used that if  $f \in \mathcal{S}(\mathbb{R}^d)$ , then  $(-\Delta)^{-\beta/2}(f) \in \mathcal{S}(\mathbb{R}^d)$ . From equation (3.6), and Lemma 2(i),

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{\gamma(2\beta)} \|\mathbf{z}\|^{-d+2\beta} \overline{f(\mathbf{z})} d\mathbf{z} = \int_{\mathbb{R}^d} \|\mathbf{z}\|^{-2\beta} \overline{\mathcal{F}(f)(\mathbf{z})} d\mathbf{z} \\
&= \int_{\mathbb{R}^d} \frac{1}{[\gamma(\beta)]^2} \left[ \int_{\mathbb{R}^d} \|\mathbf{z} - \mathbf{y}\|^{-d+\beta} \|\mathbf{y}\|^{-d+\beta} d\mathbf{y} \right] \overline{f(\mathbf{z})} d\mathbf{z}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad 0 < \beta < d/2.
\end{aligned}
\tag{3.7}$$

Let us now consider on the space of infinitely differentiable functions with compact support contained in  $D$ ,  $C_0^\infty(D) \subset \mathcal{S}(\mathbb{R}^d)$ , the norm

$$\begin{aligned}
\|f\|_{(-\Delta)^{\alpha-d/2}}^2 &= \left\langle (-\Delta)^{\alpha-d/2}(f), f \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle (-\Delta)^{\alpha-d/2}(f), f \right\rangle_{L^2(D)} \\
&= \int_{\mathbb{R}^d} (-\Delta)^{\alpha-d/2}(f)(\mathbf{x}) \overline{f(\mathbf{x})} d\mathbf{x} = \int_{\mathbb{R}^d} \frac{1}{\gamma(d-2\alpha)} \int_{\mathbb{R}^d} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} f(\mathbf{y}) \overline{f(\mathbf{x})} d\mathbf{y} d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathcal{F}(f)(\boldsymbol{\lambda})|^2 \|\boldsymbol{\lambda}\|^{-(d-2\alpha)} d\boldsymbol{\lambda}, \quad \forall f \in C_0^\infty(D), \quad 0 < \alpha < d/2.
\end{aligned}
\tag{3.8}$$

The associated inner product is given by

$$\begin{aligned}
\langle f, g \rangle_{(-\Delta)^{\alpha-d/2}} &= \int_{\mathbb{R}^d} \frac{1}{\gamma(d-2\alpha)} \int_{\mathbb{R}^d} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} f(\mathbf{y}) \overline{g(\mathbf{x})} d\mathbf{y} d\mathbf{x} \\
&= \int_D \frac{1}{\gamma(d-2\alpha)} \int_D \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} f(\mathbf{y}) \overline{g(\mathbf{x})} d\mathbf{y} d\mathbf{x},
\end{aligned}
\tag{3.9}$$

for all  $f, g \in C_0^\infty(D)$ . The closure of  $C_0^\infty(D)$  with the norm  $\|\cdot\|_{(-\Delta)^{\alpha-d/2}}$ , introduced in (3.8), defines a Hilbert space, which will be denoted as  $\mathcal{H}_{2\alpha-d} = \overline{C_0^\infty(D)}^{\|\cdot\|_{(-\Delta)^{\alpha-d/2}}}$ .

**Remark 2.** For a bounded open domain  $D$ , from Proposition 2.2. in [9], with  $D = n - 1$ ,  $p = q = 2$ , and  $s = 0$  (hence,  $A_{pq}^s(D) = A_{22}^0(D) = L^2(D)$ , where, as usual,  $L^2(D)$  denotes the space of square integrable functions on  $D$ ), we have

$$\overline{C_0^\infty(D)}^{\|\cdot\|_{L^2(\mathbb{R}^d)}} = L^2(D),
\tag{3.10}$$

(see also [45], for the case of regular bounded open domains with  $C^\infty$ -boundaries). In addition, for all  $f \in C_0^\infty(D)$ , by definition of the norm (3.8),

$$\|f\|_{(-\Delta)^{\alpha-d/2}} \leq C \|f\|_{L^2(\mathbb{R}^d)},$$

that is, all convergent sequences of  $C_0^\infty(D)$  in the  $L^2(\mathbb{R}^d)$  norm are also convergent in the  $\mathcal{H}_{2\alpha-d}$  norm. Hence, the closure of  $C_0^\infty(D)$ , with respect to the norm  $\|\cdot\|_{L^2(\mathbb{R}^d)}$ , is included in the closure of  $C_0^\infty(D)$ , with respect to the norm  $\|\cdot\|_{(-\Delta)^{\alpha-d/2}}$ . Therefore, from equation (3.10),

$$L^2(D) = \overline{C_0^\infty(D)}^{\|\cdot\|_{L^2(\mathbb{R}^d)}} \subseteq \overline{C_0^\infty(D)}^{\|\cdot\|_{(-\Delta)^{\alpha-d/2}}} = \mathcal{H}_{2\alpha-d}.
\tag{3.11}$$

The asymptotic order of the eigenvalues of operator  $\mathcal{K}_\alpha$  on  $L^2(D)$ , in the case  $d \geq 2$ , are given in the next result (see, for example, [47], [49] and [51], p.197). (See also [13] and [48], for the case  $d = 1$ ).

**Theorem 1.** *Let us consider the integral operator  $\mathcal{K}_\alpha$  introduced in equation (3.1) as an operator on the space  $L^2(D)$ , with  $D$  being a bounded open domain of  $\mathbb{R}^d$ . The following asymptotic is satisfied by the eigenvalues  $\lambda_k(\mathcal{K}_\alpha)$ ,  $k \geq 1$ , of operator  $\mathcal{K}_\alpha$  :*

$$(3.12) \quad \lim_{k \rightarrow \infty} \frac{\lambda_k(\mathcal{K}_\alpha)}{k^{-(d-\alpha)/d}} = \tilde{c}(d, \alpha) |D|^{(d-\alpha)/d},$$

where  $|D|$  denotes, as before, the Lebesgue measure of domain  $D$ , and

$$(3.13) \quad \tilde{c}(d, \alpha) = \pi^{\alpha/2} \left( \frac{2}{d} \right)^{(d-\alpha)/d} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \left[\Gamma\left(\frac{d}{2}\right)\right]^{(d-\alpha)/d}}.$$

*Proof.* We apply the results derived in [49], on the asymptotic behavior of the eigenvalues associated with certain class of integral equations. Specifically, the following integral equation is considered in that paper:

$$(3.14) \quad \int V^{1/2}(\mathbf{x}) k(\mathbf{x} - \mathbf{y}) V^{1/2}(\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \lambda f(\mathbf{x}),$$

where  $k$  is an integrable function over a Euclidean space  $E_d$  of dimension  $d$ , having positive Fourier transform, and where  $V$  is a bounded non-negative function with bounded support. In particular, [49] considers the case where  $E_d = \mathbb{R}^d$ ,  $V$  is the indicator function of a bounded domain  $D \subseteq \mathbb{R}^d$ , and  $k(\|\mathbf{x} - \mathbf{y}\|) = \|\mathbf{x} - \mathbf{y}\|^\alpha$ , for  $\alpha > -d$ , and  $\alpha \neq 0, 2, 4, \dots$ . Function  $k$  coincides in  $\mathbb{R}^d \setminus D$  with a function whose Fourier transform  $f(\boldsymbol{\xi})$  is asymptotically equal to

$$2^{d-\alpha} \pi^{d/2} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} |\boldsymbol{\xi}|^{-d+\alpha}$$

(see also the right-hand side of equation (3.4) for  $\beta = d - \alpha$ , with  $0 < \alpha < d$ ). For  $\alpha > -d$ ,  $\alpha \neq 0, 2, 4, \dots$ , the following asymptotic of the eigenvalues of the integral operator with kernel  $k(\|\mathbf{x} - \mathbf{y}\|) = \|\mathbf{x} - \mathbf{y}\|^\alpha$  is given in equation (2) in [49]:

$$(3.15) \quad \lambda_k \sim \pi^{-\alpha/2} \left( \frac{2}{d} \right)^{\frac{d+\alpha}{d}} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(\frac{-\alpha}{2}\right) \left[\Gamma\left(\frac{d}{2}\right)\right]^{(d+\alpha)/d}} \left[ \int_{\mathbb{R}^d} [V(\mathbf{x})]^{d/(d+\alpha)} d\mathbf{x} \right]^{(d+\alpha)/d} k^{-(d+\alpha)/d},$$

with

$$\int_{\mathbb{R}^d} [V(\mathbf{x})]^{d/(d-\alpha)} d\mathbf{x} = |D|.$$

Since function  $k$  in [49] coincides with the kernel of the integral operator  $\mathcal{K}_\alpha$  in equation (3.1), for  $\alpha \in (-d, 0)$ , equation (3.15) then leads to the following asymptotic of the eigenvalues of  $\mathcal{K}_\alpha$  :

$$\lambda_k(\mathcal{K}_\alpha) \sim \pi^{\alpha/2} \left( \frac{2}{d} \right)^{\frac{d-\alpha}{d}} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \left[\Gamma\left(\frac{d}{2}\right)\right]^{(d-\alpha)/d}} \left[ \int_{\mathbb{R}^d} [V(\mathbf{x})]^{d/(d-\alpha)} d\mathbf{x} \right]^{(d-\alpha)/d} k^{-(d-\alpha)/d}.$$

□

**Remark 3.** *Similar results to those ones presented in Theorem 3.2 of [48] can be derived for the spectral zeta function of the Dirichlet Laplacian on a bounded closed multidimensional interval of  $\mathbb{R}^d$  (see also [13], for the case of  $d = 1$ ). For a continuous function of the negative Dirichlet Laplacian, the explicit computation of its trace cannot always be obtained in a general regular bounded open domain of  $\mathbb{R}^d$ . Specifically, the knowledge of the eigenvalues is guaranteed for highly symmetric regions like the sphere, or regions bounded by parallel planes (see, for example, [30]; [31]; [32]). In particular, for the torus  $\mathbb{T}^2$  in  $\mathbb{R}^2$ , the Spectral Zeta Function can be explicitly computed (see, for example, [6], Chapter 1, equation (1.49), pp. 28-29).*



The following condition is assumed to be satisfied by the slowly varying function  $\mathcal{L}$  in Theorem 2 below.

**Condition A2.** For every  $m \geq 2$ , there exists a constant  $C > 0$  such that

$$(3.16) \quad \int_D \dots(m). \int_D \frac{\mathcal{L}(T\|\mathbf{x}_1 - \mathbf{x}_2\|)}{\mathcal{L}(T)\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha} \frac{\mathcal{L}(T\|\mathbf{x}_2 - \mathbf{x}_3\|)}{\mathcal{L}(T)\|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha} \cdots \frac{\mathcal{L}(T\|\mathbf{x}_m - \mathbf{x}_1\|)}{\mathcal{L}(T)\|\mathbf{x}_m - \mathbf{x}_1\|^\alpha} d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_m \leq \\ \leq C \int_D \dots(m). \int_D \frac{d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_m}{\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha \|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha \cdots \|\mathbf{x}_m - \mathbf{x}_1\|^\alpha}.$$

Note that **Condition A2** is satisfied by slowly varying functions such that

$$(3.17) \quad \sup_{T, \mathbf{x}_1, \mathbf{x}_2 \in D} \frac{\mathcal{L}(T\|\mathbf{x}_1 - \mathbf{x}_2\|)}{\mathcal{L}(T)} \leq C_0,$$

for  $0 < C_0 \leq 1$ . This condition holds for bounded slowly varying functions as in (1.6), in the case where  $D \subseteq \mathcal{B}_1(\mathbf{0})$ , with  $\mathcal{B}_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d, \|\mathbf{x}\| \leq 1\}$ .

For the derivation of the limit distribution, when  $T \rightarrow \infty$ , of the functional (1.1), we first compute its variance, in terms of  $H_2$ , the Hermite polynomial of order 2. It is well-known that Hermite polynomials form a complete orthogonal system of the Hilbert space  $L_2(\mathbb{R}, \varphi(u)du)$ , the space of square integrable functions with respect to the standard normal density  $\varphi$ . They are defined as follows:

$$H_k(u) = (-1)^k e^{\frac{u^2}{2}} \frac{d^k}{du^k} e^{-\frac{u^2}{2}}, \quad k = 0, 1, \dots$$

In particular, for a zero-mean Gaussian random field  $Y$ , for  $k \geq 1$ ,

$$(3.18) \quad \mathbb{E} H_k(Y(\mathbf{x})) = 0, \quad \mathbb{E} (H_k(Y(\mathbf{x})) H_m(Y(\mathbf{y}))) = \delta_{m,k} m! (\mathbb{E}[Y(\mathbf{x})Y(\mathbf{y})])^m$$

(see, for example, [33]).

We use some ideas from [17], Sections 1.4, 1.5 and 2.1). Consider the uniform distribution on  $D(T)$  with the density:

$$(3.19) \quad P_{D(T)}(\mathbf{x}) = T^{-d} |D|^{-1} \mathbb{I}_{\mathbf{x} \in D(T)}, \quad \mathbf{x} \in \mathbb{R}^d,$$

where  $\mathbb{I}_{\mathbf{x} \in D(T)}$  denotes the indicator function of set  $D(T)$ .

Let  $\mathbf{U}$  and  $\mathbf{V}$  be two independent and uniformly distributed inside the set  $D(T)$  random vectors. We denote  $\psi_{D(T)}(\rho)$ , the density of the Euclidean distance  $\|\mathbf{U} - \mathbf{V}\|$ . Note that  $\psi_{D(T)}(\rho) = 0$ , if  $\rho > \text{diam}(D(T))$ , and  $\psi_{D(1)}(\rho)$  is bounded, where  $\text{diam}(D(T))$  is the diameter of the set  $D(T)$ .

Using the above notation, we obtain

$$(3.20) \quad \int_{D(T)} \int_{D(T)} G(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} = |D(T)|^2 \mathbb{E}[G(\|\mathbf{U} - \mathbf{V}\|)] \\ = |D|^2 T^{2d} \int_0^{\text{diam}(D(T))} G(\rho) \psi_{D(T)}(\rho) d\rho,$$

for any Borel function  $G$  such that the Lebesgue integral (3.20) exists. In particular, under **Conditions A1–A2** for  $0 < \alpha < d/2$ , and  $T \rightarrow \infty$ , we obtain

$$(3.21) \quad \sigma^2(T) = \text{Var} \left[ \int_{D(T)} H_2(Y(\mathbf{x})) d\mathbf{x} \right] = 2 \int_{D(T)} \int_{D(T)} \frac{\mathcal{L}^2(\|\mathbf{x} - \mathbf{y}\|)}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{x} d\mathbf{y} \\ = 2! |D|^2 T^{2d} \int_0^{\text{diam}(D(T))} \mathcal{L}^2(\rho) \rho^{-2\alpha} \psi_{D(T)}(\rho) d\rho.$$

In equation (3.21), consider the change of variable  $u = \rho/T$ . Applying the consistency of the uniform distribution with a homothetic transformation, and the asymptotic properties of slowly varying functions (see Theorem 2.7 in [38]) we get

$$\begin{aligned} \sigma^2(T) &= 2|D|^2 T^{2d-2\alpha} \int_0^{\text{diam}(D)} u^{-2\alpha} \mathcal{L}^2(uT) \psi_D(u) du \\ (3.22) \quad &= |D|^2 T^{2d-2\alpha} \mathcal{L}^2(T) [a_d(D)]^2 (1 + o(1)), \quad 0 < \alpha < d/2, \quad T \rightarrow \infty, \end{aligned}$$

where, by (3.20),

$$(3.23) \quad a_d(D) = \left[ 2 \int_0^{\text{diam}(D)} u^{-2\alpha} \psi_D(u) du \right]^{1/2} = \left[ 2 \int_D \int_D \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} \right]^{1/2}.$$

More details, including properties of slowly varying functions, can be found in [3].

If  $D$  is the ball  $\mathcal{B}_T(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq T\}$ , then (see [17], Lemma 1.4.2)

$$(3.24) \quad \psi_{\mathcal{B}_T(\mathbf{0})}(\rho) = T^{-d} I_{1 - (\frac{\rho}{2T})^2} \left( \frac{d+1}{2}, \frac{1}{2} \right) d\rho, \quad 0 \leq \rho \leq 2T,$$

where

$$(3.25) \quad I_\mu(p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\mu t^{p-1} (1-t)^{q-1} dt, \quad \mu \in [0, 1], \quad p > 0, \quad q > 0,$$

is the incomplete beta function. In this case, one can show (see Lemma 2.1.3 in [17])

$$(3.26) \quad a_d(B_1(\mathbf{0})) = \frac{2^{d-2\alpha+2} \pi^{d-1/2} \Gamma\left(\frac{d-2\alpha+1}{2}\right)}{(d-2\alpha) \Gamma\left(\frac{d}{2}\right) \Gamma(d-\alpha+1)}.$$

For  $d = 1$ ,  $D = [0, 1]$ ,

$$a_1([0, 1]) = 2 \int_0^1 \int_0^1 \frac{dx dy}{|x - y|^{2\alpha}} = \frac{1}{(1-\alpha)(1-2\alpha)}, \quad 0 < \alpha < 1/2.$$

Let us now consider domain  $D = [0, l_1] \times \cdots \times [0, l_d] \subset \mathbb{R}^d$ ,  $l_i > 0$ ,  $i = 1, \dots, d$ . The Dirichlet negative Laplacian operator on such a domain has eigenvectors  $\{\phi_k\}_{k \geq 1}$  and eigenvalues  $\{\lambda_k(-\Delta)\}_{k \geq 1}$  given by

$$\begin{aligned} \phi_k(\mathbf{x}) &= \prod_{i=1}^d \sin\left(\frac{\pi k_i x_i}{l_i}\right), \quad \mathbf{x} = (x_1, \dots, x_d) \in [0, l_1] \times \cdots \times [0, l_d], \quad k_i \geq 1, \quad i = 1, \dots, d \\ \lambda_{\mathbf{k}}(-\Delta) &= \sum_{i=1}^d \frac{\pi^2 k_i^2}{l_i^2}, \quad \mathbf{k} = (k_1, \dots, k_d), \quad k_i \geq 1, \quad i = 1, \dots, d. \end{aligned} \quad (3.27)$$

The norm  $\|\mathcal{K}_\alpha^2\|_{\mathcal{L}(L^2(D))}$  of  $\mathcal{K}_\alpha^2$  in  $\mathcal{L}(L^2(D))$ , the space of bounded linear operators on  $L^2(D)$ , is given by the supremum of its eigenvalues. From Theorem 4.5(ii) in [10],

$$(3.28) \quad \lambda_k(\mathcal{K}_\alpha^2) \leq 2 \left[ \sum_{i=1}^d \frac{\pi^2 k_i^2}{l_i^2} \right]^{\alpha-d}, \quad d - \alpha \in (0, 1).$$

Note that Theorem 4.5(ii) in [10] holds for a bounded convex domain in  $\mathbb{R}^d$ . From equation (3.28), if  $\min\{l_1, \dots, l_d\} \leq \sqrt{\frac{d}{2}}\pi$ ,

$$(3.29) \quad \|\mathcal{K}_\alpha^2\|_{\mathcal{L}(L^2(D))} \leq 1.$$

**Theorem 2.** *Let  $D$  be a regular bounded open domain. Assume that **Conditions A1** and **A2** are satisfied. The following assertions then hold:*

(i) As  $T \rightarrow \infty$ , the functional  $S_T$  in (1.1) converges in distribution sense to a zero-mean random variable  $S_\infty$ , with characteristic function given by

$$(3.30) \quad \psi(z) = E[\exp(izS_\infty)] = \exp\left(\frac{1}{2} \sum_{m=2}^{\infty} \frac{(2iz)^m}{m} c_m\right), \quad z \in \mathbb{R},$$

where, for  $m \geq 2$ ,

$$(3.31) \quad \begin{aligned} c_m &= \text{Tr}(\mathcal{K}_\alpha^m) \\ &= \int_D \cdots \int_D \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha} \frac{1}{\|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha} \cdots \frac{1}{\|\mathbf{x}_m - \mathbf{x}_1\|^\alpha} d\mathbf{x}_1 \cdots d\mathbf{x}_m. \end{aligned}$$

(ii) The functional

$$S_T^H = \frac{1}{\mathcal{L}(T)T^{d-\alpha}} \left[ \int_{D(T)} G(Y(\mathbf{x})) d\mathbf{x} - C_0^H T^d |D| \right]$$

converges in distribution sense, as  $T \rightarrow \infty$ , to the random variable  $\frac{1}{2}C_2^H S_\infty$ , with  $S_\infty$  having characteristic function (3.30), and with  $G \in L^2(\mathbb{R}, \varphi(x)dx)$  having Hermite rank  $m = 2$ . Here,

$$\begin{aligned} C_0^H &= \int_{\mathbb{R}} G(u) H_0(u) \varphi(u) du = E[G(Y(\mathbf{x}))] \\ C_2^H &= \int_{\mathbb{R}} G(u) H_2(u) \varphi(u) du, \end{aligned}$$

respectively denote the 0th and 2th Hermite coefficients of the function  $G$ .

**Remark 4.** Note that **Condition A2** is satisfied by the slowly varying function (1.6) with  $C = 1$ , for  $D = \mathcal{B}_1(\mathbf{0}) = \{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$ .

*Proof.* We first prove (i). Since  $EY^2(\mathbf{x}) = 1$ ,

$$\int_{D(T)} d\mathbf{x} = \int_{D(T)} E[Y^2(\mathbf{x})] d\mathbf{x} = E \left[ \int_{D(T)} Y^2(\mathbf{x}) d\mathbf{x} \right] = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)}) E\eta_j^2 = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)}).$$

From Definition 1, Lemma 1, and Remark 1, one has

$$\begin{aligned} \psi_T(z) &= E \left[ \exp \left( \frac{iz}{dT} \int_{D(T)} (Y^2(\mathbf{x}) - 1) d\mathbf{x} \right) \right] \\ &= \exp \left( -\frac{iz \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})}{dT} \right) \prod_{j=1}^{\infty} \left( 1 - 2iz \frac{\lambda_{j,T}(R_{Y,D(T)})}{dT} \right)^{-1/2} \\ &= \exp \left( -\frac{iz \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})}{dT} \right) \left[ \mathcal{D}_T \left( \frac{2iz}{dT} \right) \right]^{-1/2} \\ &= \exp \left( -\frac{iz \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})}{dT} \right) \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{2iz}{dT} \right)^m \text{Tr} \left( R_{Y,D(T)}^m \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \exp \left( -\frac{iz \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})}{d_T} + \frac{iz \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})}{d_T} \right. \\
&\quad \left. + \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m} \left( \frac{2iz}{d_T} \right)^m \text{Tr} \left( R_{Y,D(T)}^m \right) \right) \\
&= \exp \left( \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m} \left( \frac{2iz}{d_T} \right)^m \text{Tr} \left( R_{Y,D(T)}^m \right) \right).
\end{aligned}
\tag{3.32}$$

Furthermore, under **Condition A2**, there exists a positive constant  $C$  such that

$$\begin{aligned}
\frac{1}{d_T^2} \text{Tr} \left( R_{Y,D(T)}^2 \right) &= \int_D \int_D \frac{\mathcal{L}(T\|\mathbf{x}_1 - \mathbf{x}_2\|)}{\mathcal{L}(T)} \frac{\mathcal{L}(T\|\mathbf{x}_2 - \mathbf{x}_1\|)}{\mathcal{L}(T)} \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_2\|^{2\alpha}} d\mathbf{x}_1 d\mathbf{x}_2 \\
&\leq C \int_D \int_D \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_2\|^{2\alpha}} d\mathbf{x}_1 d\mathbf{x}_2 = C \text{Tr} \left( \mathcal{K}_\alpha^2 \right) < \infty
\end{aligned}
\tag{3.33}$$

$$\begin{aligned}
&\frac{1}{d_T^m} \text{Tr} \left( R_{Y,D(T)}^m \right) = \\
&= \frac{1}{[\mathcal{L}(T)]^m} \int_D \dots \int_D \frac{\mathcal{L}(T\|\mathbf{x}_1 - \mathbf{x}_2\|)}{\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha} \frac{\mathcal{L}(T\|\mathbf{x}_2 - \mathbf{x}_3\|)}{\|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha} \dots \frac{\mathcal{L}(T\|\mathbf{x}_m - \mathbf{x}_1\|)}{\|\mathbf{x}_m - \mathbf{x}_1\|^\alpha} d\mathbf{x}_1 \dots d\mathbf{x}_m \\
&\leq C \int_D \dots \int_D \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha} \frac{1}{\|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha} \dots \frac{1}{\|\mathbf{x}_m - \mathbf{x}_1\|^\alpha} d\mathbf{x}_1 \dots d\mathbf{x}_m \\
&= C \text{Tr} \left( \mathcal{K}_\alpha^m \right) < \infty, \quad m > 2,
\end{aligned}
\tag{3.34}$$

since  $\|\mathcal{K}_\alpha^m\|_1 \leq M \|\mathcal{K}_\alpha^2\|_1$ , for  $m > 2$  and  $M > 0$ .

From equations (3.32)–(3.34), for every  $T > 0$ ,

$$\begin{aligned}
|\psi_T(z)| &= \left| \exp \left( \frac{1}{2} \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{2m-2} \left( \frac{2z}{d_T} \right)^{2m-2} \text{Tr} \left( R_{Y,D(T)}^{2m-2} \right) \right) \right| \\
&\times \left| \exp \left( \frac{i}{2} \sum_{m=3}^{\infty} \frac{(-1)^m}{2m-3} \left( \frac{2z}{d_T} \right)^{2m-3} \text{Tr} \left( R_{Y,D(T)}^{2m-3} \right) \right) \right| \\
&= \left| \exp \left( \frac{1}{2} \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{2m-2} \left( \frac{2z}{d_T} \right)^{2m-2} \text{Tr} \left( R_{Y,D(T)}^{2m-2} \right) \right) \right| \\
&\times \left[ \cos^2 \left( \frac{1}{2} \sum_{m=3}^{\infty} \frac{(-1)^m}{2m-3} \left( \frac{2z}{d_T} \right)^{2m-3} \text{Tr} \left( R_{Y,D(T)}^{2m-3} \right) \right) \right. \\
&\quad \left. + \sin^2 \left( \frac{1}{2} \sum_{m=3}^{\infty} \frac{(-1)^m}{2m-3} \left( \frac{2z}{d_T} \right)^{2m-3} \text{Tr} \left( R_{Y,D(T)}^{2m-3} \right) \right) \right]^{1/2}
\end{aligned}
\tag{3.35}$$

$$\begin{aligned}
&= \left| \exp \left( \frac{-1}{2} \sum_{m=2}^{\infty} \frac{(-1)^m}{2m-2} \left( \frac{2z}{d_T} \right)^{2m-2} \text{Tr} \left( R_{Y,D(T)}^{2m-2} \right) \right) \right| \\
&= \left| \exp \left( \frac{-1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{4n-2} \left( \frac{2z}{d_T} \right)^{4n-2} \text{Tr} \left( R_{Y,D(T)}^{4n-2} \right) \right) \right|
\end{aligned}
\tag{3.36}$$

$$\times \left| \exp \left( \frac{-1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{4n} \left( \frac{2z}{d_T} \right)^{4n} \text{Tr} \left( R_{Y,D(T)}^{4n} \right) \right) \right|
\tag{3.37}$$

$$\begin{aligned}
&= \left| \exp \left( \frac{-1}{2} \sum_{n=1}^{\infty} \frac{1}{4n-2} \left( \frac{2z}{d_T} \right)^{4n-2} \text{Tr} \left( R_{Y,D(T)}^{4n-2} \right) \right) \right| \\
&\times \left| \exp \left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{4n} \left( \frac{2z}{d_T} \right)^{4n} \text{Tr} \left( R_{Y,D(T)}^{4n} \right) \right) \right| \\
&\leq \left| \exp \left( \frac{-1}{2} \sum_{n=1}^{\infty} \frac{1}{4n-2} \left( \frac{2z}{d_T} \right)^{4n-2} \text{Tr} \left( R_{Y,D(T)}^{4n-2} \right) \right) \right| \\
&\times \left| \exp \left( \frac{C}{2} \sum_{n=1}^{\infty} \frac{1}{4n} (2z)^{4n} \text{Tr} (\mathcal{K}_{\alpha}^{4n}) \right) \right| \\
&\leq \left| \exp \left( \frac{C}{2} \sum_{n=1}^{\infty} \frac{1}{4n} (2z)^{4n} \text{Tr} (\mathcal{K}_{\alpha}^{4n}) \right) \right| \\
(3.38) \quad &= \left| \exp \left( \frac{C}{8} \sum_{n=1}^{\infty} \frac{1}{n} (16z^4)^n \text{Tr} ((\mathcal{K}_{\alpha}^4)^n) \right) \right| = [\mathcal{D}_{\mathcal{K}_{\alpha}^4}(16z^4)]^{-C/8} < \infty,
\end{aligned}$$

where we have applied, inside the argument of the exponential, the straightforward identities  $i^{2m-2} = (i^2)^{m-1} = (-1)^{m-1}$ ,  $|\exp(iu)| = \cos^2(u) + \sin^2(u) = 1$ , and the fact that the sequence of natural numbers  $m = 2, 3, 4, 5, 6, \dots = \mathbb{N} - \{0, 1\}$  can be obtained as the union of the sequences  $\{2m-2\}_{m \geq 2} = 2, 4, 6, \dots$  and  $\{2m-3\}_{m \geq 3} = 3, 5, 7, \dots$ . Hence, in the above equation, the sum in  $\mathbb{N} - \{0, 1\}$  can be splitted into the sums  $\sum_{m=2}^{\infty} (-1)^{m-1} f(2m-2)$  and  $\sum_{m=3}^{\infty} (-1)^m f(2m-3)$ . Moreover, in (3.37), we consider the sequence  $\{2m-2\}_{m \geq 2} = 2, 4, 6, 8, 10, 12, \dots$  as the union of the sequences  $\{4n-2\}_{n \geq 1} = 2, 6, 10, \dots$ , and  $\{4n\}_{n \geq 1} = 4, 8, 12, \dots$ , corresponding to the changes of variable  $m = 2n$  and  $m = 2n+1$ . Thus, for  $m = 2n$ ,  $2m-2 = 4n-2$ , and, for  $m = 2n+1$ ,  $2m-2 = 4n+2-2 = 4n$ . The sum  $\sum_{m=2}^{\infty} (-1)^m f(2m-2)$  can then be splitted into the two sums  $\sum_{n=1}^{\infty} (-1)^{2n} f(4n-2)$  and  $\sum_{n=1}^{\infty} (-1)^{2n+1} f(4n)$ . Furthermore, the last identity in (3.38) is obtained from the Fredholm determinant formula

$$(3.39) \quad \mathcal{D}_{\mathcal{K}_{\alpha}^4} = \det(I - \omega \mathcal{K}_{\alpha}^4) = \exp \left( - \sum_{k=1}^{\infty} \frac{\text{Tr}[\mathcal{K}_{\alpha}^4]^k}{k} \omega^k \right) = \exp \left( - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [\lambda_l(\mathcal{K}_{\alpha}^4)]^k \frac{\omega^k}{k} \right)$$

of  $\mathcal{K}_{\alpha}^4$  at point  $\omega = 16z^4$ , which is finite for  $|\omega| < \frac{1}{\|\mathcal{K}_{\alpha}^4\|_1}$ , i.e., for  $|z| < \frac{1}{4\|\mathcal{K}_{\alpha}^4\|_1^{1/4}}$ , since, from Theorem 1 (see also equations (3.1) and (3.23)),

$$(3.40) \quad \text{Tr}(\mathcal{K}_{\alpha}^2) = \int_D \int_D \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{x} d\mathbf{y} = \frac{[a_d(D)]^2}{2} < \infty.$$

Hence,  $\|\mathcal{K}_{\alpha}^4\|_1 \leq \widetilde{M} \|\mathcal{K}_{\alpha}^2\|_1 = \widetilde{M} \text{Tr}(\mathcal{K}_{\alpha}^2) < \infty$ , for certain  $\widetilde{M} > 0$ .

From (3.38), there exists  $\tilde{\psi}(z) = \lim_{T \rightarrow \infty} |\psi_T(z)| < \infty$ , for  $0 < z < [1/16\|\mathcal{K}_{\alpha}^4\|_1]^{1/4}$ . An analytic continuation argument (see [26], Th. 7.1.1) guarantees that  $\tilde{\psi}$  defines the unique limit characteristic function for all real values of  $z$ .

From (3.38), we now prove that  $\tilde{\psi}(z) = \psi(z)$ , with  $\psi(z)$  given in (3.30)–(3.31).

Consider, for each  $z \in \mathbb{R}$ , the sequence

$$(3.41) \quad \left\{ \frac{1}{m} \left( \frac{2iz}{d_T} \right)^m \text{Tr} \left( R_{Y,D(T)}^m \right) \right\}_{m \geq 2},$$

involved in the series appearing in the argument of the complex exponential defining  $\psi_T(z)$ , in equation (3.32). Keeping in mind that  $(i)^{2m} = (-1)^m$ , and that  $(i)^{2m+1} = (-1)^m i$ , let us split

(3.41) into the subsequences:

$$(3.42) \quad \{P_m\}_{m \geq 1} = \left\{ \frac{(-1)^m}{2m} \left( \frac{2z}{dT} \right)^{2m} \text{Tr} \left( R_{Y,D(T)}^{2m} \right) \right\}_{m \geq 1}$$

$$(3.43) \quad \{Q_m\}_{m \geq 1} = \left\{ \frac{(-1)^m i}{2m+1} \left( \frac{2z}{dT} \right)^{2m+1} \text{Tr} \left( R_{Y,D(T)}^{2m+1} \right) \right\}_{m \geq 1}.$$

Under **Condition A2**, the following two inequalities hold:

$$(3.44) \quad \left| \frac{(-1)^m}{2m} \left( \frac{2z}{dT} \right)^{2m} \text{Tr} \left( R_{Y,D(T)}^{2m} \right) \right| \leq C \frac{1}{2m} (2|z|)^{2m} \text{Tr} (\mathcal{K}_\alpha^{2m})$$

$$(3.45) \quad \left| \frac{(-1)^m i}{2m+1} \left( \frac{2z}{dT} \right)^{2m+1} \text{Tr} \left( R_{Y,D(T)}^{2m+1} \right) \right| \leq C \frac{1}{2m+1} (2|z|)^{2m+1} \text{Tr} (\mathcal{K}_\alpha^{2m+1}).$$

Moreover, for  $|z| < 1/2$ ,

$$(3.46) \quad \begin{aligned} \sum_{m=1}^{\infty} C \frac{1}{2m} (2|z|)^{2m} \text{Tr} (\mathcal{K}_\alpha^{2m}) &\leq \sum_{m=1}^{\infty} \frac{C}{m} (2|z|)^m \text{Tr} (\mathcal{K}_\alpha^{2m}) \\ &= \ln \left( [\mathcal{D}_{\mathcal{K}_\alpha^2}(2|z|)]^{-C} \right) < \infty \end{aligned}$$

$$(3.47) \quad \begin{aligned} \sum_{m=1}^{\infty} \frac{C}{2m+1} (2|z|)^{2m+1} \text{Tr} (\mathcal{K}_\alpha^{2m+1}) &\leq \sum_{m=1}^{\infty} \frac{C}{m} (2|z|)^m M \text{Tr} (\mathcal{K}_\alpha^{2m}) \\ &= \ln \left( [\mathcal{D}_{\mathcal{K}_\alpha^2}(2|z|)]^{-CM} \right) < \infty, \end{aligned}$$

considering  $|z| < \frac{1}{2} \wedge \frac{1}{2\text{Tr}(\mathcal{K}_\alpha^2)}$ , where Definition 1 of the Fredholm determinant of  $\mathcal{K}_\alpha^2$  has been applied, since, as commented before,  $\mathcal{K}_\alpha^2$  is in the trace class from Theorem 1. Here,  $M$  is a positive constant such that, for every  $m \geq 1$ ,  $\text{Tr} (\mathcal{K}_\alpha^{2m+1}) \leq M \text{Tr} (\mathcal{K}_\alpha^{2m})$ , in the case where  $\|\mathcal{K}_\alpha^2\|_{\mathcal{L}(L^2(D))} \leq 1$ . Otherwise,

$$\begin{aligned} \text{Tr} (\mathcal{K}_\alpha^{2m+1}) &\leq M' \text{Tr} (\mathcal{K}_\alpha^{2m+2}) \\ &\leq M' \left[ \sup_{k \geq 1} \lambda_k(\mathcal{K}_\alpha^2) \right]^2 \text{Tr} (\mathcal{K}_\alpha^{2m}). \end{aligned}$$

Hence  $M = M' \left[ \sup_{k \geq 1} \lambda_k(\mathcal{K}_\alpha^2) \right]^2$ .

Inequalities (3.46) and (3.47) allow us to apply Dominated Convergence Theorem for integration with respect to a counting measure obtaining, for  $|z| < \frac{1}{2} \wedge \frac{1}{2\text{Tr}(\mathcal{K}_\alpha^2)}$ ,

$$(3.48) \quad \begin{aligned} \lim_{T \rightarrow \infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} \left( \frac{2z}{dT} \right)^{2m} \text{Tr} \left( R_{Y,D(T)}^{2m} \right) &= \sum_{m=1}^{\infty} \lim_{T \rightarrow \infty} \frac{(-1)^m}{2m} \left( \frac{2z}{dT} \right)^{2m} \text{Tr} \left( R_{Y,D(T)}^{2m} \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} (2z)^{2m} \text{Tr} (\mathcal{K}_\alpha^{2m}) \end{aligned}$$

$$(3.49) \quad \begin{aligned} \lim_{T \rightarrow \infty} \sum_{m=1}^{\infty} \frac{(-1)^m i}{2m+1} \left( \frac{2z}{dT} \right)^{2m+1} \text{Tr} \left( R_{Y,D(T)}^{2m+1} \right) &= \sum_{m=1}^{\infty} \lim_{T \rightarrow \infty} \frac{(-1)^m i}{2m+1} \left( \frac{2z}{dT} \right)^{2m+1} \text{Tr} \left( R_{Y,D(T)}^{2m+1} \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m i}{2m+1} (2z)^{2m+1} \text{Tr} (\mathcal{K}_\alpha^{2m+1}), \end{aligned}$$

where we have considered that, from (1.3), for every  $m \geq 2$ ,

$$(3.50) \quad \lim_{T \rightarrow \infty} \frac{\text{Tr} \left( R_{Y,D(T)}^m \right)}{d_T^m} = \text{Tr} \left( \mathcal{K}_\alpha^m \right),$$

which is finite from Theorem 1, that ensures the trace property of  $\mathcal{K}_\alpha^2$ . Hence, for every  $m \geq 2$ , and  $z$ ,

$$(3.51) \quad \lim_{T \rightarrow \infty} \frac{1}{m} \left( \frac{2iz}{d_T} \right)^m \text{Tr} \left( R_{Y,D(T)}^m \right) = \frac{1}{m} (2iz)^m \text{Tr} \left( \mathcal{K}_\alpha^m \right) = \frac{1}{m} (2iz)^m c_m,$$

with  $c_m$  being given in equation (3.31).

From equations (3.32), (3.42), (3.43), (3.48), (3.49) and (3.51), we obtain, for  $|z| < \frac{1}{2} \wedge \frac{1}{2\text{Tr}(\mathcal{K}_\alpha^2)}$ ,

$$(3.52) \quad \begin{aligned} \lim_{T \rightarrow \infty} \psi_T(z) &= \lim_{T \rightarrow \infty} \exp \left( \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m} \left( \frac{2iz}{d_T} \right)^m \text{Tr} \left( R_{Y,D(T)}^m \right) \right) \\ &= \lim_{T \rightarrow \infty} \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} P_m \right) \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} Q_m \right) \\ &= \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \lim_{T \rightarrow \infty} P_m \right) \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \lim_{T \rightarrow \infty} Q_m \right) \\ &= \exp \left( \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m} (2iz)^m \text{Tr} \left( \mathcal{K}_\alpha^m \right) \right) = \psi(z). \end{aligned}$$

An analytic continuation argument (see [26], Th. 7.1.1) guarantees that  $\psi$  defines the unique limit characteristic function for all real values of  $z$ .

We now turn to the proof of (ii). Under **Condition A1**, since  $B(\|\mathbf{x}\|) \leq 1$ , and  $B(0) = 1$ , we have

$$B^j(\|\mathbf{x}\|) \leq B^3(\|\mathbf{x}\|), \quad j \geq 3.$$

Hence,

$$(3.53) \quad \begin{aligned} K_T &= \left[ \frac{1}{\mathcal{L}^2(T)T^{2d-2\alpha}} \right] E \left[ \left( \int_{D(T)} G(Y(\mathbf{x})) d\mathbf{x} - C_0^H T^d |D| - \frac{C_2^H}{2} \int_{D(T)} H_2(Y(\mathbf{x})) d\mathbf{x} \right) \right]^2 \\ &= \left[ \frac{1}{\mathcal{L}^2(T)T^{2d-2\alpha}} \right] \sum_{j=3}^{\infty} \frac{(C_j^H)^2}{j!} \int_{D(T)} \int_{D(T)} B^j(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} \leq \\ &\leq \left[ \frac{1}{\mathcal{L}^2(T)T^{2d-2\alpha}} \right] \int_{D(T)} \int_{D(T)} B^3(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} \left[ \sum_{j=3}^{\infty} \frac{(C_j^H)^2}{j!} \right]. \end{aligned}$$

By **Condition A1**, for any  $\varepsilon > 0$ , there exists  $A_0 > 0$ , such that for  $\|\mathbf{x} - \mathbf{y}\| > A_0$ ,  $B(\|\mathbf{x} - \mathbf{y}\|) < \varepsilon$ . Let  $\mathcal{D}_1 = \{(\mathbf{x}, \mathbf{y}) \in D(T) \times D(T) : \|\mathbf{x} - \mathbf{y}\| \leq A_0\}$ ,  $\mathcal{D}_2 = \{(\mathbf{x}, \mathbf{y}) \in D(T) \times D(T) : \|\mathbf{x} - \mathbf{y}\| > A_0\}$ ,

$$(3.54) \quad \int_{D(T)} \int_{D(T)} B^3(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} = \left\{ \int \int_{\mathcal{D}_1} + \int \int_{\mathcal{D}_2} \right\} B^3(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} = S_T^{(1)} + S_T^{(2)}.$$

Using the bound  $B^3(\|\mathbf{x} - \mathbf{y}\|) \leq 1$  on  $\mathcal{D}_1$ , and the bound  $B^3(\|\mathbf{x} - \mathbf{y}\|) < \epsilon B^2(\|\mathbf{x} - \mathbf{y}\|)$  on  $\mathcal{D}_2$ , we obtain,

$$|S_T^{(1)}| \leq \int \int_{\mathcal{D}_1} |B^3(\|\mathbf{x} - \mathbf{y}\|)| d\mathbf{x} d\mathbf{y} \leq M_1 T^d$$

for a suitable constant  $M_1 > 0$ , and for  $T$  sufficiently large, under **A2**,

$$\begin{aligned} |S_T^{(2)}| &\leq \int \int_{\mathcal{D}_2} |B^3(\|\mathbf{x} - \mathbf{y}\|)| d\mathbf{x} d\mathbf{y} \leq \epsilon \int \int_{\mathcal{D}_2} B^2(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} \\ &\leq \epsilon \int_{D(T)} \int_{D(T)} B^2(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} = \epsilon \int_{D(T)} \int_{D(T)} \frac{\mathcal{L}^2(\|\mathbf{x} - \mathbf{y}\|)}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{x} d\mathbf{y} \\ &= \epsilon \mathcal{L}^2(T) T^{2d-2\alpha} \int_{D(1)} \int_{D(1)} \frac{\mathcal{L}^2(T\|\mathbf{x} - \mathbf{y}\|)}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha} \mathcal{L}^2(T)} d\mathbf{x} d\mathbf{y} \\ (3.55) \quad &\leq \epsilon C \mathcal{L}^2(T) T^{2d-2\alpha} \int_{D(1)} \int_{D(1)} \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} < \infty, \quad 0 < \alpha < d/2. \end{aligned}$$

Thus, for

$$M_2 = C \int_{D(1)} \int_{D(1)} \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}},$$

from (3.53)–(3.55), we have

$$\begin{aligned} K_T &\leq \left[ \frac{1}{\mathcal{L}(T) T^{d-\alpha}} \right]^2 \left[ \sum_{j=3}^{\infty} \frac{(C_j^H)^2}{j!} \right] \int_{D(T)} \int_{D(T)} B^3(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} \\ (3.56) \quad &\leq (M_1 \vee M_2) \left[ \frac{T^d}{\mathcal{L}^2(T) T^{2d-2\alpha}} + \epsilon \frac{T^{2d-2\alpha} \mathcal{L}^2(T)}{\mathcal{L}^2(T) T^{2d-2\alpha}} \right] \end{aligned}$$

is arbitrarily small together with  $\epsilon > 0$  as  $T \rightarrow \infty$ . The desired result on weak-convergence then follows.  $\square$

**Remark 5.** Note that the proof of Theorem 2(i) is slightly different from the proof of Theorem 2 in [23], for the case of non-linear functionals of chi-squared random fields. Specifically, we use here similar arguments to those ones considered in that paper, to derive the existence of a limit. However, to obtain the explicit limit characteristic function, we apply Dominated Convergence Theorem in a different way, considering equations (3.41)–(3.43).

**Remark 6.** Consider the case of  $d = 1$  and discrete time. That is, let  $\{Y(t), t \in \mathbb{Z}\}$  be a stationary zero-mean Gaussian sequence with unit variance and covariance function of the form

$$B(t) = \frac{\mathcal{L}(t)}{|t|^\alpha},$$

for  $0 < \alpha < 1/2$ . The proof of the weak convergence result in [35] and [42] is based on the following formula for the characteristic function of a quadratic form of strong-correlated Gaussian random variables:

$$\begin{aligned} E \left[ \exp \left\{ i z \frac{1}{d_T} \sum_{t=0}^{T-1} (Y^2(t) - 1) \right\} \right] &= \exp \{ -i z T d_T^{-1} \} \left[ \det (I_T - 2i z d_T^{-1} R_T) \right]^{-1/2} \\ (3.57) \quad &= \exp \left\{ \sum_{k=2}^{\infty} (2i z d_T^{-1})^k \frac{Sp R_T^k}{k} \right\}, \end{aligned}$$



where

$$(3.58) \quad \frac{1}{d_T^k} \text{Sp} R_T^k = \frac{1}{d_T^k} \sum_{i_1=0}^{T-1} \cdots \sum_{i_k=0}^{T-1} B(|i_1 - i_2|) B(|i_2 - i_3|) \cdots B(|i_k - i_1|),$$

with  $d_T = T^{1-\alpha} \mathcal{L}(T)$ ,  $R_T = E[Y\bar{Y}']$ ,  $Y = (Y(0), \dots, Y(T-1))'$ ,  $\text{Sp} R_T$  denoting the trace of the matrix  $R_T$ , and  $I_T$  representing the identity matrix of size  $T$  (see p. 39 of [29]). One can get a direct extension of formulae (3.57) and (3.58) to the stationary zero-mean Gaussian random process case in continuous time  $\{Y(t), t \in \mathbb{R}\}$  (see [24]), but for  $d \geq 2$  direct extensions of (3.57) and (3.58) are not available. The present paper addresses this problem by applying alternative functional tools, like the Karhunen-Loève expansion and Fredholm determinant formula, to overcome this difficulty of discretization of the multidimensional parameter space. Note that the Fredholm determinant formula appears in the definition of the characteristic functional of quadratic forms defined in terms of Hilbert-valued zero-mean Gaussian random variables (see, for example, Proposition 1.2.8 in [11]).

**Remark 7.** Expanding around zero the characteristic function (3.30), we obtain the cumulants of random variable  $S_\infty$ , that is,  $\kappa_1 = 0$ , and

$$(3.59) \quad \kappa_k = 2^{k-1} (k-1)! c_k, \quad k \geq 2,$$

where  $c_k$  are defined as in equation (3.31). The derivation of explicit expressions for  $c_k$  would lead to the computation of the moments or cumulants of the limit distribution. This aspect will constitute the subject of a subsequent paper.

#### 4. INFINITE SERIES REPRESENTATION AND EIGENVALUES

The representation of the Rosenblatt-type distribution as the sum of an infinite series of weighted independent chi-squared random variables is derived in this section. As in the classical case (see Proposition 2 in [12]), this series expansion is obtained from the double Wiener-Itô stochastic integral representation of  $S_\infty$  in the spectral domain (see Theorem 3). Proposition 1 and Corollary 2 below establish the connection between the eigenvalues of operator  $\mathcal{K}_\alpha$  in (3.1) and the weights appearing in the series representation derived.

The following condition will be required for the derivation of Theorem 3(ii) below.

**Condition A3.** Suppose that **Condition A1** holds, and there exists a spectral density  $f_0(\|\lambda\|)$ ,  $\lambda \in \mathbb{R}^d$ , being decreasing function for  $\|\lambda\| \in (0, \varepsilon]$ , with  $\varepsilon > 0$ .

If **Condition A3** holds, from equation (1.4), applying a Tauberian Theorem (see [14], and Theorems 4 and 11 in [22]),

$$(4.1) \quad f_0(\|\lambda\|) \sim c(d, \alpha) \mathcal{L} \left( \frac{1}{\|\lambda\|} \right) \|\lambda\|^{\alpha-d}, \quad 0 < \alpha < d, \quad \|\lambda\| \rightarrow 0.$$

Here,  $c(d, \alpha) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2})}$  is defined in (3.3).

**Condition A3** holds, in particular, for the correlation function (1.5), with the isotropic spectral density

$$(4.2) \quad f_0(\|\lambda\|) = \frac{\|\lambda\|^{1-\frac{d}{2}}}{2^{\frac{d}{2}-1} \pi^{\frac{d}{2}+1}} \int_0^\infty K_{\frac{d}{2}-1}(\|\lambda\|u) \frac{\sin \left( \gamma \arg \left( 1 + u^\beta \exp \left( \frac{i\pi\beta}{2} \right) \right) \right)}{\left| 1 + u^\beta \exp \left( \frac{i\pi\beta}{2} \right) \right|^\gamma} u^{\frac{d}{2}} du,$$

where  $K_\nu(z)$  is the modified Bessel function of the second kind. By Corollary 3.10 in [25], the spectral density (4.2) satisfies (4.1), with  $\alpha = \beta\gamma < d$ .

The zero-mean Gaussian random field  $Y$  with an absolutely continuous spectrum has the isonormal representation

$$(4.3) \quad Y(\mathbf{x}) = \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_0(\|\boldsymbol{\lambda}\|)} Z(d\boldsymbol{\lambda}),$$

where  $Z$  is a complex white noise Gaussian random measure with Lebesgue control measure.

**Theorem 3.** *Let  $D$  be a regular bounded open domain.*

(i) *For  $0 < \alpha < d/2$ , the following identities hold:*

$$(4.4) \quad \int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{(\|\boldsymbol{\lambda}_1\| \|\boldsymbol{\lambda}_2\|)^{d-\alpha}} = \left[ \frac{a_d \gamma(\alpha)}{\sqrt{2}|D|} \right]^2 = \frac{[\gamma(\alpha)]^2 \text{Tr}(\mathcal{K}_\alpha^2)}{|D|^2} < \infty,$$

where  $a_d$  is defined in (3.23),  $\gamma(\alpha)$  is introduced in equation (3.3), and  $K$  is the characteristic function of the uniform distribution over set  $D$ , given by

$$(4.5) \quad K(\boldsymbol{\lambda}, D) = \int_D e^{i\langle \boldsymbol{\lambda}, \mathbf{x} \rangle} p_D(\mathbf{x}) d\mathbf{x} = \frac{1}{|D|} \int_D e^{i\langle \boldsymbol{\lambda}, \mathbf{x} \rangle} d\mathbf{x} = \frac{\vartheta(\boldsymbol{\lambda})}{|D|},$$

with associated probability density function  $p_D(\mathbf{x}) = 1/|D|$  if  $\mathbf{x} \in D$ , and 0 otherwise.

(ii) *Assume that **Conditions A1, A2, A3** hold. Then, the random variable  $S_\infty$  admits the following double Wiener-Itô stochastic integral representation:*

$$(4.6) \quad S_\infty = |D|c(d, \alpha) \int_{\mathbb{R}^{2d}}'' H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \frac{Z(d\boldsymbol{\lambda}_1) Z(d\boldsymbol{\lambda}_2)}{\|\boldsymbol{\lambda}_1\|^{\frac{d-\alpha}{2}} \|\boldsymbol{\lambda}_2\|^{\frac{d-\alpha}{2}}},$$

where  $Z$  is a complex white noise Gaussian measure with Lebesgue control measure, and the notation  $\int_{\mathbb{R}^{2d}}''$  means that one does not integrate on the hyperdiagonals  $\boldsymbol{\lambda}_1 = \pm \boldsymbol{\lambda}_2$ . Here, the kernel  $H$  is given by:

$$(4.7) \quad H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D),$$

$$\text{and } c(d, \alpha) = \frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)} = \frac{1}{\gamma(\alpha)}.$$

**Remark 8.** *Our goal in this paper is to focus on the case of Hermite rank  $m = 2$ , which has very special properties not shared by the higher orders, such as the existence of eigenvalues. We are aware of the extension to all Hermite ranks, as described, for example, in the more general and different approach presented in the monograph by Major (1981).*

*Proof.* (i) From equation (3.8) and the proof of Theorem 1,

$$(4.8) \quad \begin{aligned} \|1_D\|_{\mathcal{H}_{2\alpha-d}}^2 &= \int_D \frac{1}{\gamma(d-2\alpha)} \int_D \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\ &= \frac{a_d^2}{2\gamma(d-2\alpha)} = \frac{1}{\gamma(d-2\alpha)} \sum_{j=1}^{\infty} \lambda_j^2(\mathcal{K}_\alpha^2) = \frac{\text{Tr}(\mathcal{K}_\alpha^2)}{\gamma(d-2\alpha)} < \infty, \end{aligned}$$

since  $\mathcal{K}_\alpha^2$  is in the trace class. Therefore,  $1_D$  belongs to the Hilbert space  $\mathcal{H}_{2\alpha-d}$  with the inner product introduced in equation (3.9). From equation (3.8), we then obtain

$$\frac{a_d^2}{2\gamma(d-2\alpha)} = \|1_D\|_{\mathcal{H}_{2\alpha-d}}^2 = \frac{|D|^2}{(2\pi)^d} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, D)|^2 \|\boldsymbol{\omega}_1\|^{-(d-2\alpha)} d\boldsymbol{\omega}_1.$$

It is well-known that the Fourier transform defines an automorphism on the Schwartz space, which, in particular, contains  $C_0^\infty(D)$ . Thus, the Fourier transform of any function in the space

$\mathcal{H}_{2\alpha-d}$  can be defined as the limit in the space  $\mathcal{H}_{2\alpha-d}$  of the Fourier transforms of functions in  $C_0^\infty(D)$ . Therefore, from equation (3.7) with  $f(\mathbf{z}) = |D|^2 |K(\mathbf{z}, D)|^2$ ,

$$\begin{aligned} \frac{a_d^2}{2\gamma(d-2\alpha)} &= \|1_D\|_{\mathcal{H}_{2\alpha-d}}^2 = \frac{|D|^2}{(2\pi)^d} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, D)|^2 \|\boldsymbol{\omega}_1\|^{-d+2\alpha} d\boldsymbol{\omega}_1 \\ &= \frac{|D|^2}{(2\pi)^d} \frac{\gamma(2\alpha)}{[\gamma(\alpha)]^2} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, D)|^2 \left[ \int_{\mathbb{R}^d} \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|^{-d+\alpha} \|\boldsymbol{\omega}_2\|^{-d+\alpha} d\boldsymbol{\omega}_2 \right] d\boldsymbol{\omega}_1 \\ &= \frac{|D|^2 \gamma(2\alpha)}{(2\pi)^d [\gamma(\alpha)]^2} \int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{(\|\boldsymbol{\lambda}_1\| \|\boldsymbol{\lambda}_2\|)^{d-\alpha}}. \end{aligned}$$

Hence,

$$(4.9) \quad \frac{a_d^2}{2} = \left[ \frac{|D|}{\gamma(\alpha)} \right]^2 \int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{(\|\boldsymbol{\lambda}_1\| \|\boldsymbol{\lambda}_2\|)^{d-\alpha}},$$

since  $\frac{\gamma(2\alpha)\gamma(d-2\alpha)}{(2\pi)^d} = 1$ . Note that, we also have applied the fact that, from Remark 2,

$$1_D \star 1_D(\mathbf{x}) = \int_{\mathbb{R}^d} 1_D(\mathbf{y}) 1_D(\mathbf{x} + \mathbf{y}) d\mathbf{y} = \int_D 1_D(\mathbf{x} + \mathbf{y}) d\mathbf{y} \in L^2(D) \subseteq \mathcal{H}_{2\alpha-d},$$

since

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} 1_D(\mathbf{y}) 1_D(\mathbf{x} + \mathbf{y}) d\mathbf{y} \right|^2 d\mathbf{x} \leq |\mathcal{B}_{R(D)}(\mathbf{0})|^3,$$

where, as before,  $|\mathcal{B}_{R(D)}|$  denotes the Lebesgue measure of the ball of center  $\mathbf{0}$  and radius  $R(D)$ , with  $R(D)$  being equal to two times the diameter of the regular bounded open set  $D$  containing the point  $\mathbf{0}$ . Hence,  $\mathcal{F}(1_D \star 1_D)(\boldsymbol{\lambda}) = |D|^2 |K(\boldsymbol{\lambda}, D)|^2$  belongs to the space of Fourier transforms of functions in  $\mathcal{H}_{2\alpha-d}$ . Summarizing, equation (4.9) provides the finiteness of (4.4), i.e., assertion (i) holds due to the trace property of  $\mathcal{K}_\alpha^2$  for the regular bounded domain  $D$  considered (see Theorem 1).

(ii) The proof of this part of Theorem 3 can be obtained as a particular case of Theorem 5 in [22] (see also Remark 6 in that paper). Note that convexity is not used in the proof of Theorem 5 of [22]. An outline of the proof of Theorem 5 in [22] for the case of Hermite rank equal to two is now given.

Under **Conditions A1, A3** (see also (4.3)),

$$(4.10) \quad Y(\mathbf{x}) = \frac{|D(T)|}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) K(\boldsymbol{\lambda}, D(T)) f_0^{1/2}(\boldsymbol{\lambda}) Z(d\boldsymbol{\lambda}), \quad \mathbf{x} \in D(T).$$

Using the self-similarity of Gaussian white noise, and the Itô formula (see, for example, [12]; [28]), we obtain from equation (4.10)

$$\begin{aligned} S_T &= \frac{1}{T^{d-\alpha} \mathcal{L}(T)} \int_{D(T)} H_2(Y(\mathbf{x})) d\mathbf{x} \\ &= \frac{c(d, \alpha) |D(T)|}{T^{d-\alpha} \mathcal{L}(T)} \int_{\mathbb{R}^{2d}} K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D(T)) \left( \frac{1}{c(d, \alpha)} \prod_{j=1}^2 f_0^{1/2}(\boldsymbol{\lambda}_j) \right) Z(d\boldsymbol{\lambda}_1) Z(d\boldsymbol{\lambda}_2) \\ &= \frac{c(d, \alpha) |D|}{T^{d-\alpha} \mathcal{L}(T)} \int_{\mathbb{R}^{2d}} K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D) \left( \frac{1}{c(d, \alpha)} \prod_{j=1}^2 f_0^{1/2}(\boldsymbol{\lambda}_j/T) \right) Z(d\boldsymbol{\lambda}_1) Z(d\boldsymbol{\lambda}_2). \end{aligned} \quad (4.11)$$

By the isometry property of multiple stochastic integrals

$$(4.12) \quad \begin{aligned} & \mathbb{E} \left[ S_T - c(d, \alpha) |D| \int_{\mathbb{R}^{2d}}'' H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \frac{Z(d\boldsymbol{\lambda}_1) Z(d\boldsymbol{\lambda}_2)}{\|\boldsymbol{\lambda}_1\|^{\frac{d-\alpha}{2}} \|\boldsymbol{\lambda}_2\|^{\frac{d-\alpha}{2}}} \right]^2 = \\ & = \int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)|^2 [c(d, \alpha) |D|]^2 Q_T(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{\|\boldsymbol{\lambda}_1\|^{d-\alpha} \|\boldsymbol{\lambda}_2\|^{d-\alpha}}, \end{aligned}$$

where

$$(4.13) \quad Q_T(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \left( \left[ \frac{\|\boldsymbol{\lambda}_1\|^{(d-\alpha)/2} \|\boldsymbol{\lambda}_2\|^{(d-\alpha)/2}}{T^{d-\alpha} \mathcal{L}(T) c(d, \alpha)} \prod_{j=1}^2 f_0^{1/2}(\boldsymbol{\lambda}_j/T) \right] - 1 \right)^2.$$

From equation (4.1), under **Condition A3**, we obtain the pointwise convergence of  $Q_T(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$  to 0, as  $T \rightarrow \infty$ . By Lebesgue's Dominated Convergence Theorem, the integral converges to zero if there is some integrable function which dominates integrands for all T. This fact can be proved as in pp. 21–22 of [22], applying previous assertion (i) derived in this theorem.  $\square$

Alternatively, in the proof of Theorem 3(ii), the class  $\tilde{\mathcal{LC}}$  of slowly varying functions, introduced in Definition 9 in [21], can also be considered. Note that an infinitely differentiable function  $\mathcal{L}(\cdot)$  belongs to the class  $\tilde{\mathcal{LC}}$  if

1. for any  $\delta > 0$ , there exists  $\lambda_0(\delta) > 0$  such that  $\lambda^{-\delta} \mathcal{L}(\lambda)$  is decreasing and  $\lambda^\delta \mathcal{L}(\lambda)$  is increasing if  $\lambda > \lambda_0(\delta)$ ;
2.  $\mathcal{L}_j \in \mathcal{SL}$ , for all  $j \geq 0$ , where  $\mathcal{L}_0(\lambda) := \mathcal{L}$ ,  $\mathcal{L}_{j+1}(\lambda) := \lambda \mathcal{L}'_j(\lambda)$ , with  $\mathcal{SL}$  being the class of functions that are slowly varying at infinity and bounded on each finite interval.

In that case, the following lemma can be applied in the proof of Theorem 3(ii).

**Lemma 3.** *Let  $\alpha \in (0, d)$ ,  $S \in C^\infty(s_{n-1}(1))$ , and  $\mathcal{L} \in \tilde{\mathcal{LC}}$ . Let  $\{\xi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$  be a mean-square continuous homogeneous random field with zero mean. Let the field  $\xi(\mathbf{x})$  has the spectral density  $f_0(\mathbf{u})$ ,  $\mathbf{u} \in \mathbb{R}^d$ , which is infinitely differentiable for all  $\mathbf{u} \neq 0$ . If the covariance function  $B(\mathbf{x})$ ,  $\mathbf{x} \in \mathbb{R}^d$ , of the field has the following behavior*

- (a)  $\|\mathbf{x}\|^\alpha B(\mathbf{x}) \sim S\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \mathcal{L}(\|\mathbf{x}\|)$ ,  $\mathbf{x} \rightarrow \infty$ ,  
the spectral density satisfies the condition
- (b)  $\|\mathbf{u}\|^{d-\alpha} f_0(\mathbf{u}) \sim \tilde{S}_{\alpha,d}\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) \mathcal{L}\left(\frac{1}{\|\mathbf{u}\|}\right)$ ,  $\|\mathbf{u}\| \rightarrow 0$ .

On the other hand, from Theorems 1 and 3(i), the spectral asymptotics of  $\mathcal{K}_\alpha$  and the Dirichlet Laplacian operator on  $L^2(D)$  can be applied to verifying the finiteness of (4.4) for a wide class of domains  $D$ . Drum and fractal drum are two families of well-known regular compact sets where Weyl's classical theorem on the asymptotic behavior of the eigenvalues has been extended (see, for example, [16]; [20]; [46]). In particular, as illustration of Theorem 3(i), we now refer to the case of regular compact domains constructed from the finite union of convex compact sets like balls, or by their difference which is the case, for instance, of circular rings.

*Examples.* Let  $D = \mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2, 0)) \subset \mathbb{R}^2$ , with  $\mathcal{B}_1(\mathbf{0}) = \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq 1\}$ , and  $\mathcal{B}_1((2, 0)) = \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1 - 2)^2 + x_2^2} \leq 1\}$ . It is well-known (see [17], p. 57, Lemma 2.1.3) that, for  $\mathcal{B}_1(\mathbf{0}) \subset \mathbb{R}^2$  and  $0 < \alpha < 1$ ,

$$Tr([\mathcal{K}_\alpha^{\mathcal{B}_1(\mathbf{0})}]^2) = \int_{\mathcal{B}_1(\mathbf{0})} \int_{\mathcal{B}_1(\mathbf{0})} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} = \frac{2^{2-2\alpha+1} \pi^{2-\frac{1}{2}} \Gamma(\frac{2-2\alpha+1}{2})}{(2-2\alpha) \Gamma(2-\alpha+1)},$$

where, to avoid confusion, for a subset  $S$ , we have used the notation  $\mathcal{K}_\alpha^S$  to represent operator  $\mathcal{K}_\alpha$  acting on the space  $L^2(S)$ , and  $[\mathcal{K}_\alpha^S]^2 = \mathcal{K}_\alpha^S \mathcal{K}_\alpha^S$ .

Hence,

$$\begin{aligned}
& \int_{\mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0))} \int_{\mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0))} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\
& \leq \int_{\mathcal{B}_3(\mathbf{0})} \int_{\mathcal{B}_3(\mathbf{0})} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\
& = \text{Tr} \left( [\mathcal{K}_\alpha^{\mathcal{B}_3(\mathbf{0})}]^2 \right) = 3^{4-2\alpha} \text{Tr} \left( [\mathcal{K}_\alpha^{\mathcal{B}_1(\mathbf{0})}]^2 \right) = \frac{3^{4-2\alpha} 2^{2-2\alpha+1} \pi^{2-\frac{1}{2}} \Gamma(\frac{2-2\alpha+1}{2})}{(2-2\alpha)\Gamma(2-\alpha+1)} < \infty.
\end{aligned}
\tag{4.14}$$

From Theorem 3(i), equation (4.14) provides the finiteness of (4.4) for non-convex compact set  $D = \mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0))$ .

These computations can be easily extended to the finite union of balls with the same or with different radius, and to the case  $d > 2$ , considering the value of the integral

$$\int_{\mathcal{B}_R(\mathbf{0})} \int_{\mathcal{B}_R(\mathbf{0})} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} = R^{2d-2\alpha} a_d(\mathcal{B}_1(\mathbf{0})) \frac{1}{2},$$

where the constant  $a_d(\mathcal{B}_1(\mathbf{0}))$  is defined in (3.26), for  $0 < \alpha < d/2$  (see [17], p. 57, Lemma 2.1.3).

For the case of a circular ring, that is, for

$$D = \mathcal{B}_{R_1}(\mathbf{0}) \setminus \mathcal{B}_{R_2}(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : R_2 < \|\mathbf{x}\| < R_1\}, \quad R_1 > R_2 > 0,$$

we can proceed in a similar way to the above-considered example. Specifically,

$$\begin{aligned}
& \int_{\mathcal{B}_{R_1}(\mathbf{0}) \setminus \mathcal{B}_{R_2}(\mathbf{0})} \int_{\mathcal{B}_{R_1}(\mathbf{0}) \setminus \mathcal{B}_{R_2}(\mathbf{0})} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\
& \leq \int_{\mathcal{B}_{R_1}(\mathbf{0})} \int_{\mathcal{B}_{R_1}(\mathbf{0})} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\
& = \text{Tr} \left( [\mathcal{K}_\alpha^{\mathcal{B}_{R_1}(\mathbf{0})}]^2 \right) = R_1^{4-2\alpha} \text{Tr} \left( [\mathcal{K}_\alpha^{\mathcal{B}(\mathbf{0})}]^2 \right) = \frac{R_1^{4-2\alpha} 2^{2-2\alpha+1} \pi^{2-\frac{1}{2}} \Gamma(\frac{2-2\alpha+1}{2})}{(2-2\alpha)\Gamma(2-\alpha+1)} < \infty.
\end{aligned}$$

From Theorem 3(i), equation (4.4) is finite for  $D = \mathcal{B}_{R_1}(\mathbf{0}) \setminus \mathcal{B}_{R_2}(\mathbf{0})$ . Similarly, these computations can be extended to the finite union of circular rings.

**Remark 9.** Note that for a ball  $D = \mathcal{B}_1(\mathbf{0}) = \mathcal{B}(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d; \|\mathbf{x}\| \leq 1\}$ , the function  $\vartheta(\boldsymbol{\lambda})$  in (4.5) is of the form  $\int_{\mathcal{B}_1(\mathbf{0})} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) d\mathbf{x} = (2\pi)^{d/2} \frac{\mathcal{J}_{d/2}(\|\boldsymbol{\lambda}\|)}{\|\boldsymbol{\lambda}\|^{d/2}}$ , for  $d \geq 2$ , where  $\mathcal{J}_\nu(\mathbf{z})$  is the Bessel function of the first kind and order  $\nu > -1/2$ . For a rectangle,  $D = \prod = \{a_i \leq x_i \leq b_i, i = 1, \dots, d\}$ , with  $\mathbf{0} \in \prod$ , we have  $\vartheta(\boldsymbol{\lambda}) = \prod_{j=1}^d (\exp(i\lambda_j b_j) - \exp(i\lambda_j a_j)) / i\lambda_j$ , for  $d \geq 1$ . Moreover for  $d = 2$ , considering the non-convex set  $D = \mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0)) \subset \mathbb{R}^2$ ,

$$\vartheta(\boldsymbol{\lambda}) = \vartheta(\lambda_1, \lambda_2) = \int_{\mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0))} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) d\mathbf{x} = \frac{2\pi \mathcal{J}_1(\|\boldsymbol{\lambda}\|)}{\|\boldsymbol{\lambda}\|} (1 + \exp(2i\lambda_1)),$$

and for  $D = \mathcal{B}_{R_1}(\mathbf{0}) \setminus \mathcal{B}_{R_2}(\mathbf{0})$ ,  $\vartheta(\boldsymbol{\lambda}) = (2\pi R_1) \mathcal{J}_1(\|\boldsymbol{\lambda}\| R_1) / \|\boldsymbol{\lambda}\| - (2\pi R_2) \mathcal{J}_1(\|\boldsymbol{\lambda}\| R_2) / \|\boldsymbol{\lambda}\|$ .

The following corollary is an extension of Proposition 2 in [12].

**Corollary 1.** Assume that the conditions of Theorem 3 hold. Then, the limit random variable  $S_\infty$  admits the following series representation:

$$(4.15) \quad S_\infty = \frac{1}{d} c(d, \alpha) |D| \sum_{\mathbf{n} \in \mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H}) (\varepsilon_{\mathbf{n}}^2 - 1) = \sum_{\mathbf{n} \in \mathbb{N}_*^d} \lambda_{\mathbf{n}}(S_\infty) (\varepsilon_{\mathbf{n}}^2 - 1),$$

where

$$c(d, \alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)} = \frac{1}{\gamma(\alpha)}$$

was already introduced in (3.3),  $\varepsilon_{\mathbf{n}}$  are independent and identically distributed standard Gaussian random variables, and  $\mu_{\mathbf{n}}(\mathcal{H})$ ,  $\mathbf{n} \in \mathbb{N}_*^d$ , is a sequence of non-negative real numbers, which are the eigenvalues of the self-adjoint Hilbert-Schmidt operator

$$(4.16) \quad \mathcal{H}(h)(\lambda_1) = \int_{\mathbb{R}^d} H_1(\lambda_1 - \lambda_2) h(\lambda_2) G_\alpha(d\lambda_2) : L_E^2(\mathbb{R}^d, G_\alpha) \longrightarrow L_E^2(\mathbb{R}^d, G_\alpha),$$

with

$$(4.17) \quad G_\alpha(d\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^{d-\alpha}} d\mathbf{x},$$

and  $L_E^2(\mathbb{R}^d, G_\alpha)$  denoting the collection of linear combinations, with real-valued coefficients, of complex-valued and Hermitian functions, that are square integrable with respect to  $G_\alpha(d\mathbf{x})$ . Note that  $L_E^2(\mathbb{R}^d, G_\alpha)$  is a real Hilbert space, endowed with the scalar product

$$\langle \psi_1, \psi_2 \rangle_{G_\alpha} = \int_{\mathbb{R}^d} \psi_1(\mathbf{x}) \overline{\psi_2(\mathbf{x})} G_\alpha(d\mathbf{x})$$

(see [33], pp. 159-161, for the case of  $L_E^2(\mathbb{R}, d\beta)$  spaces). The symmetric kernel  $H_1(\lambda_1 - \lambda_2) = H(\lambda_1, \lambda_2)$ , is defined from  $H$  introduced in equation (4.7), in terms of the characteristic function  $K$  given in equation (4.5).

The proof can be seen in Appendix A. Indeed, it constitutes an extension of Proposition 2 in [12].

In the following proposition the explicit relationship between the eigenvalues of  $\mathcal{K}_\alpha$  and  $\mathcal{H}$  is derived. Note that Theorem 3(i) provides the equality between the traces of operators  $\frac{\mathcal{K}_\alpha^2}{[|D|c(d, \alpha)]^2}$  and  $\mathcal{H}^2$ , with, as before,  $\mathcal{H}$  having kernel  $H(\cdot, \cdot)$  given in equation (4.7).

**Proposition 1.** *The operators  $\mathcal{A}_\alpha : L_E^2(\mathbb{R}^d, G_\alpha) \longrightarrow L_E^2(\mathbb{R}^d, G_\alpha)$*

$$\mathcal{A}_\alpha(f)(\lambda_1) = c(d, \alpha) \int_{\mathbb{R}^d} H_1(\lambda_1 - \lambda_2) f(\lambda_2) G_\alpha(d\lambda_2),$$

and  $|D|^{-1} \mathcal{K}_\alpha : L^2(D) \longrightarrow L^2(D)$  have the same eigenvalues. Here,  $c(d, \alpha)$  was already introduced in (3.3),  $H_1(\lambda_1 - \lambda_2) = H(\lambda_1, \lambda_2)$  with kernel  $H$  being given in equation (4.7),  $G_\alpha$  is introduced in (4.17), and  $\mathcal{K}_\alpha$  is defined in (3.1).

The proof of this result is also given in Appendix A. (See [48], for the case  $d = 1$ ).

**Corollary 2.** *Let  $\{\lambda_k(S_\infty), k \geq 1\}$  be the eigenvalues appearing in representation (4.15), arranged into a decreasing order of the magnitudes of their modulus. Then, Theorem 1 holds for this system of eigenvalues.*

The proof directly follows from Corollary 1, Proposition 1 and Theorem 1.

## 5. PROPERTIES OF ROSENBLATT-TYPE DISTRIBUTION

This section provides the Lévy-Khintchine representation of the limit random variable  $S_\infty$  (see [48], for  $d = 1$ , in the discrete time case), as well as its membership to a subclass of selfdecomposable distributions, given by the Thorin class. The absolute continuity of the law of  $S_\infty$ , and the boundedness of its probability density is then obtained.

It is well-known that the distribution of a random variable  $X$  is infinitely divisible if for any integer  $n \geq 1$ , there exist  $X_j^{(n)}$ ,  $j = 1, 2, \dots, n$ , independent and identically distributed (i.i.d.) random variables such that  $X = \sum_{j=1}^n X_j^{(n)}$ . Let  $\mathcal{ID}(\mathbb{R})$  be the class of infinitely divisible distributions or random variables. Recall that the cumulant function of an infinitely divisible random variable  $X$  admits the Lévy-Khintchine representation

$$(5.1) \quad \log E[\exp(i\theta X)] = ia\theta - \frac{b}{2}\theta^2 + \int_{-\infty}^{\infty} (\exp(i\theta u) - 1 - i\tau(u)\theta)\mu(du), \quad \theta \in \mathbb{R},$$

where  $a \in \mathbb{R}$ ,  $b \geq 0$ , and

$$(5.2) \quad \tau(u) = \begin{cases} u & |u| \leq 1 \\ \frac{u}{|u|} & |u| > 1, \end{cases}$$

and where the Lévy measure  $\mu$  is a Radon measure on  $\mathbb{R} \setminus \{0\}$  such that  $\mu(\{0\}) = 0$  and

$$\int \min(u^2, 1)\mu(du) < \infty.$$

An infinitely divisible random variable  $X$  (or its law) is selfdecomposable if its characteristic function  $\phi(\theta) = E[\exp(i\theta X)]$ ,  $\theta \in \mathbb{R}$ , has the property that for every  $c \in (0, 1)$  there exists a characteristic function  $\phi_c$  such that  $\phi(\theta) = \phi(c\theta)\phi_c(\theta)$ ,  $\theta \in \mathbb{R}$ . It is known (see [37], p.95, Corollary 15.11) that an infinitely divisible law is selfdecomposable if its Lévy measure has a density  $q$  satisfying

$$q(u) = \frac{h(u)}{|u|}, \quad u \in \mathbb{R},$$

with  $h(u)$  being increasing on  $(-\infty, 0)$  and decreasing on  $(0, \infty)$ . Let  $\mathcal{SD}(\mathbb{R})$  be the class of selfdecomposable distributions or random variables. If  $Y \in \mathcal{SD}(\mathbb{R})$  then (see [19])

$$(5.3) \quad Y \stackrel{d}{=} \int_0^\infty \exp(-s) dZ(s) \stackrel{d}{=} \int_0^\infty \exp(-s\lambda) dZ(s\lambda), \quad \lambda > 0,$$

where  $\{Z(t), t \geq 0\}$  is a Lévy process whose law is determined by that of  $Y$ .

We next define the Thorin class on  $\mathbb{R}$  (see [44]; [7]; [18]) as follows: We refer to  $\gamma x$  as an *elementary gamma random variable* if  $x$  is nonrandom non-zero vector in  $\mathbb{R}$ , and  $\gamma$  is a gamma random variable on  $\mathbb{R}_+$ . Then, the Thorin class on  $\mathbb{R}$  (or the class of extended generalized gamma convolutions), denoted by  $T(\mathbb{R})$ , is defined as the smallest class of distributions that contains all elementary gamma distributions on  $\mathbb{R}$ , and is closed under convolution and weak convergence. It is known that  $T(\mathbb{R}) \subset \mathcal{SD}(\mathbb{R}) \subset \mathcal{ID}(\mathbb{R})$ , and inclusions are strict. Since any selfdecomposable distribution on  $\mathbb{R}$  is absolutely continuous (see, for instance, Example 27.8 in [37]) and is unimodal (by [50]; see also Theorem 53.1 in [37]), then, any selfdecomposable distribution has a bounded density function.

If a probability distribution function  $F$  belongs to  $T(\mathbb{R})$ , then, its characteristic function has the form (see [44], [7])

$$(5.4) \quad \phi(\theta) = \exp\left(i\theta a - \frac{b\theta^2}{2} - \int_{\mathbb{R}} \left[\log\left(1 - \frac{i\theta}{u}\right) + \frac{i\theta}{1+u^2}\right] U(du)\right),$$

where  $a \in \mathbb{R}$ ,  $b \geq 0$ , and  $U(du)$  is a non-decreasing measure on  $\mathbb{R} \setminus \{0\}$ , called Thorin measure, such that

$$U(0) = 0, \quad \int_{-1}^1 |\log |u|| U(du) < \infty, \quad \int_{-\infty}^{-1} \frac{1}{u^2} U(du) + \int_1^{\infty} \frac{1}{u^2} U(du) < \infty.$$

The Lévy density of a distribution from Thorin class is such that

$$(5.5) \quad |u|q(u) = \begin{cases} \int_0^{\infty} \exp(-yu) U(dy), & u > 0 \\ \int_0^{\infty} \exp(yu) U(dy), & u < 0, \end{cases}$$

where  $U(du)$  is the Thorin measure. In other words, the Lévy density is of the form  $h(|u|)/|u|$ , where  $h(|u|) = h_0(r)$ ,  $r \geq 0$ , is a completely monotone function over  $(0, \infty)$ .

The following result establishes the Lévy-Khintchine representation of  $S_{\infty}$ , as well as the asymptotic orders at zero and at infinity of its associated Lévy density. The membership to the Thorin self-decomposable subclass is then obtained. As a direct consequence, we then have the existence and boundedness of the probability density of  $S_{\infty}$  (see, for instance, Example 27.8 in [37]).

**Theorem 4.** *Let  $S_{\infty}$  be given as in Theorem 2 with  $0 < \alpha < d/2$ . Let us consider  $\lambda_k(S_{\infty})$ ,  $k \geq 1$ , the sequence of eigenvalues introduced in Corollary 1 satisfying the properties stated in Theorem 1 (see Corollary 2). Then,*

(i)  $S_{\infty} \in \mathcal{ID}(\mathbb{R})$  with the following Lévy-Khintchine representation:

$$(5.6) \quad \phi(\theta) = E[i\theta S_{\infty}] = \exp \left( \int_0^{\infty} (\exp(iu\theta) - 1 - iu\theta) \mu_{\alpha/d}(du) \right),$$

where  $\mu_{\alpha/d}$  is supported on  $(0, \infty)$  having density

$$(5.7) \quad q_{\alpha/d}(u) = \frac{1}{2u} \sum_{k=1}^{\infty} \exp \left( -\frac{u}{2\lambda_k(S_{\infty})} \right), \quad u > 0.$$

Furthermore,  $q_{\alpha/d}$  has the following asymptotics as  $u \rightarrow 0^+$  and  $u \rightarrow \infty$ ,

$$(5.8) \quad \begin{aligned} q_{\alpha/d}(u) &\sim \frac{[\tilde{c}(d, \alpha) |D|^{(d-\alpha)/d}]^{1/(1-\alpha/d)} \Gamma \left( \frac{1}{1-\alpha/d} \right) \left( \frac{u}{2} \right)^{-1/(1-\alpha/d)}}{2u[(1-\alpha/d)]} \\ &= \frac{2^{\frac{\alpha/d}{1-\alpha/d}} [\tilde{c}(d, \alpha) |D|^{(d-\alpha)/d}]^{1/(1-\alpha/d)} \Gamma \left( \frac{1}{1-\alpha/d} \right) u^{\frac{(\alpha/d)-2}{(1-\alpha/d)}}}{[(1-\alpha/d)]} \quad \text{as } u \rightarrow 0^+, \\ q_{\alpha/d}(u) &\sim \frac{1}{2u} \exp(-u/2\lambda_1(S_{\infty})), \quad \text{as } u \rightarrow \infty, \end{aligned}$$

where  $\tilde{c}(d, \alpha)$  is defined as in equation (3.13).

(ii)  $S_{\infty} \in \mathcal{SD}(\mathbb{R})$ , and hence it has a bounded density.

(iii)  $S_{\infty} \in T(\mathbb{R})$ , with Thorin measure given by

$$U(dx) = \frac{1}{2} \sum_{k=1}^{\infty} \delta_{\frac{1}{2\lambda_k(S_{\infty})}}(x),$$

where  $\delta_a(x)$  is the Dirac delta-function at point  $a$ .

(iv)  $S_{\infty}$  admits the integral representation

$$(5.9) \quad S_{\infty} \stackrel{d}{=} \int_0^{\infty} \exp(-u) d \left( \sum_{k=1}^{\infty} \lambda_k(S_{\infty}) A^{(k)}(u) \right) \stackrel{d}{=} \int_0^{\infty} \exp(-u) dZ(u),$$



where

$$(5.10) \quad Z(t) = \sum_{k=1}^{\infty} \lambda_k(S_{\infty}) A^{(k)}(t), \quad t \geq 0,$$

with  $A^{(k)}$ ,  $k \geq 1$ , being independent copies of a Lévy process.

*Proof.* (i) The proof follows from Theorem 1, equation (3.12), Corollary 2, and Lemma 4 below (see Appendix B), in a similar way to Theorem 4.2 in [48]. Specifically, let us first consider a truncated version of the random series representation (4.15)

$$S_{\infty}^{(M)} = \sum_{k=1}^M \lambda_k(S_{\infty}) (\varepsilon_k^2 - 1),$$

with  $S_{\infty}^{(M)} \xrightarrow[d]{} S_{\infty}$ , as  $M$  tends to infinity. From the Lévy-Khintchine representation of the chi-square distribution (see, for instance, [5], Example 1.3.22),

$$\begin{aligned} E \left[ \exp(i\theta S_{\infty}^{(M)}) \right] &= \prod_{k=1}^M E \left[ \exp(i\theta \lambda_k(S_{\infty}) (\varepsilon_k^2 - 1)) \right] \\ &= \prod_{k=1}^M \exp \left( -i\theta \lambda_k(S_{\infty}) + \int_0^{\infty} (\exp(i\theta u) - 1) \left[ \frac{\exp(-u/(2\lambda_k(S_{\infty})))}{2u} \right] du \right) \\ &= \prod_{k=1}^M \exp \left( \int_0^{\infty} (\exp(i\theta u) - 1 - i\theta u) \left[ \frac{\exp(-u/2\lambda_k(S_{\infty}))}{2u} \right] du \right) \\ (5.11) \quad &= \exp \left( \int_0^{\infty} (\exp(i\theta u) - 1 - i\theta u) \left[ \frac{1}{2u} G_{\lambda(\alpha/d)}^{(M)}(\exp(-u/2)) \right] du \right), \end{aligned}$$

where  $G_{\lambda(\alpha/d)}^{(M)}(x) = \sum_{k=1}^M x^{[\lambda_k(S_{\infty})]^{-1}}$ . To apply the Dominated Convergence Theorem, the following upper bound is used:

$$\begin{aligned} \left| (\exp(i\theta u) - 1 - i\theta u) \left[ \frac{1}{2u} G_{\lambda(\alpha/d)}^{(M)}(\exp(-u/2)) \right] \right| &\leq \frac{\theta^2}{4} u G_{\lambda(\alpha/d)}^{(M)}(\exp(-u/2)) \\ &\leq \frac{\theta^2}{4} u G_{\lambda(\alpha/d)}(\exp(-u/2)), \end{aligned} \quad (5.12)$$

where, as indicated in [48], we have applied the inequality  $|\exp(iz) - 1 - z| \leq \frac{z^2}{2}$ , for  $z \in \mathbb{R}$ . The right-hand side of (5.12) is continuous, for  $0 < u < \infty$ , and from Theorem 1, equation (3.12), Corollary 2, and Lemma 4 in Appendix B, we obtain

$$\begin{aligned} u G_{\lambda(\alpha/d)}(\exp(-u/2)) &\sim u \exp(-u/2\lambda_1(S_{\infty})), \quad \text{as } u \rightarrow \infty \\ u G_{\lambda(\alpha/d)}(\exp(-u/2)) &\sim [\tilde{c}(d, \alpha) |D|^{1-\alpha/d}]^{1/1-\alpha/d} \frac{u}{(1-\alpha/d)} \\ &\quad \Gamma \left( \frac{1}{1-\alpha/d} \right) (1 - \exp(-u/2))^{-1/(1-\alpha/d)} \\ (5.13) \quad &\sim C u^{-\frac{\alpha/d}{1-\alpha/d}} \quad \text{as } u \rightarrow 0, \end{aligned}$$

for some constant  $C$ . Since  $0 < \frac{\alpha/d}{1-\alpha/d} < 1$ , equation (5.13) implies that the right-hand side of (5.12), which does not depend on  $M$ , is integrable on  $(0, \infty)$ . Hence, by the Dominated

Convergence Theorem, as  $M \rightarrow \infty$ ,

$$(5.14) \quad \begin{aligned} & E \left[ \exp(i\theta S_\infty^{(M)}) \right] \longrightarrow E \left[ \exp(i\theta S_\infty) \right] \\ & = \exp \left( \int_0^\infty (\exp(i\theta u) - 1 - i\theta u) \left[ \frac{1}{2u} G_{\lambda(\alpha/d)}(\exp(-u/2)) \right] du \right), \end{aligned}$$

which proves that equations (5.6) and (5.7) hold.

Again, from Theorem 1, equation (3.12), Corollary 2, and Lemma 4 below,

$$(5.15) \quad \begin{aligned} & \frac{1}{2u} G_{\lambda(\alpha/d)}(\exp(-u/2)) \sim [\tilde{c}(d, \alpha) |D|^{1-\alpha/d}]^{1/(1-\alpha/d)} \frac{\Gamma\left(\frac{1}{1-\alpha/d}\right) \left(\frac{u}{2}\right)^{-1/(1-\alpha/d)}}{2u[(1-\alpha/d)]} \\ & = \frac{2^{\frac{\alpha/d}{1-\alpha/d}} [\tilde{c}(d, \alpha) |D|^{1-\alpha/d}]^{1/(1-\alpha/d)} \Gamma\left(\frac{1}{1-\alpha/d}\right) u^{\frac{(\alpha/d)-2}{(1-\alpha/d)}}}{[(1-\alpha/d)]} \quad \text{as } u \rightarrow 0 \\ & \frac{1}{2u} G_{\lambda(\alpha/d)}(\exp(-u/2)) \sim \frac{1}{2u} \exp(-u/2\lambda_1(S_\infty)) \quad \text{as } u \rightarrow \infty. \end{aligned}$$

Thus, equation (5.15) provides the asymptotic orders given in (5.8).

(ii) From (i), it follows that  $S_\infty \in \mathcal{SD}(\mathbb{R})$ , and hence it has a bounded density (see [8], Example 27.8 in [37], and [50]). Note that an alternative proof of the boundedness of the probability density of  $S_\infty$  is provided in Appendix C, where an upper bound is also obtained.

(iii) In view of (5.5) and (5.7),  $S_\infty \in T(\mathbb{R})$  with Thorin measure given by

$$(5.16) \quad U(dx) = \frac{1}{2} \sum_{k=1}^{\infty} \delta_{\frac{1}{2\lambda_k(S_\infty)}}(x),$$

where  $\delta_a(x)$  is the Dirac delta-function at point  $a$ . From Theorem 1, Corollaries 1 and Proposition 1 (see also Corollary 2), the number of terms in the sum (5.16) is infinite. Hence, the Thorin measure  $U(dx)$ , as a counting measure, has infinite total mass. The form of Thorin measure is a direct consequence of (5.5) and (5.7).

(iv) As in [27], we consider a gamma subordinator  $\gamma_\lambda(t)$ ,  $t \geq 0$ , with parameter  $\lambda > 0$ , that is, a Lévy process such that  $\gamma_\lambda(0) = 0$ , and  $P\{\gamma_\lambda(t) \in dx\} = \lambda^{-t} \Gamma^{-1}(t) \exp(-x\lambda) x^{t-1} dx$ ,  $x > 0$ , and a homogeneous Poisson process  $N(t)$ ,  $t \geq 0$ , with unit rate. Assume that the two processes are independent. Then (see [4]), for any  $c > 0$ , and  $\lambda > 0$ , the representation (5.3) (see [19]) can be specified as follows:

$$\gamma_\lambda(c) = \int_0^\infty \exp(-t) d\gamma_\lambda(N(ct)).$$

The process  $A(t) = \gamma_{1/2}(N(t/2)) - t$ ,  $t \geq 0$ , is a Lévy process.

For  $k \geq 1$ , let us consider  $\gamma_{\frac{1}{2}}^{(k)}\left(\frac{1}{2}\right)$  and  $A^{(k)}(t)$  to be independent copies of  $\gamma_{\frac{1}{2}}\left(\frac{1}{2}\right)$  and  $A(t)$ , respectively. Then, we have

$$\varepsilon_k^2 - 1 = \gamma_{\frac{1}{2}}^{(k)}\left(\frac{1}{2}\right) = \int_0^\infty \exp(-u) dA^{(k)}(u),$$

where  $\varepsilon_k$  are independent and identically distributed standard normal random variables as given in the series expansion (4.15). Then, for  $\lambda_k(S_\infty)$ ,  $k \geq 1$ , being the eigenvalues appearing in such a series expansion, arranged into a decreasing order of their magnitudes, we obtain that the distribution of  $S_\infty$  admits the integral representation (5.9), with,  $A^k$ ,  $k \geq 1$ , in equation (5.10) being independent copies of the Lévy process  $A(t) = \gamma_{1/2}(N(t/2)) - t$ ,  $t \geq 0$ .

□

For any  $0 < \alpha/d < 1/2$ , the Lévy measure  $\mu_{\alpha/d}$  satisfies

$$\int_0^\infty u^2 \mu_{\alpha/d}(du) = \mathbb{E}[S_\infty^2] = [a_d(D)]^2.$$

Furthermore, when  $\alpha/d \rightarrow 1/2$ , since  $(\exp(i\theta u) - 1 - i\theta u) \rightarrow (-1/2)\theta^2$  (see [48]), we have

$$\phi(\theta) = \exp\left(\int_0^\infty \frac{\exp(i\theta u) - 1 - i\theta u}{u^2} u^2 \mu_{\alpha/d}(du)\right) \rightarrow \exp\left(-\frac{1}{2}\theta^2\right),$$

which means that  $S_\infty \rightarrow N(0, 1)$ .

In addition, from Theorem 4, it can be proved, in a similar way to Corollary 4.3 and 4.4 in [48], that, for  $0 < \alpha/d < 1/2$ , the probability density function of  $S_\infty$  is infinitely differentiable with all derivatives tending to 0 as  $|x| \rightarrow \infty$ . Also, the following inequality holds

$$P[S_\infty < -x] \leq \exp\left(-\frac{1}{2}x^2\right), \quad x > 0.$$

We also note that, for  $\epsilon > 0$ ,

$$\lim_{u \rightarrow \infty} \frac{P[S_\infty > u + \epsilon]}{P[S_\infty > u]} = \exp\left(-\frac{\epsilon}{2\lambda_1(S_\infty)}\right).$$

**Remark 10.** In view of the integral representation (5.9), one can construct an Ornstein-Uhlenbeck type process

$$dS(t) = -\lambda S(t) + dL(\lambda t), \quad t \geq 0, \quad \lambda > 0,$$

driven by a Lévy process  $L(t)$ ,  $t \geq 0$ , and with marginal Rosenblatt distribution  $S_\infty$ . The driving process  $L(t)$  is referred to as the background Lévy process, and it is introduced in (5.10).

## 6. APPENDICES

### APPENDIX A

**Proof of Corollary 1.** From condition (4.4), operator  $\mathcal{H}$  is a Hilbert-Schmidt operator from  $L_E^2(\mathbb{R}^d, G_\alpha)$  into  $L_E^2(\mathbb{R}^d, G_\alpha)$ , which admits a spectral decomposition, in terms of a sequence of eigenvalues  $\{\mu_{\mathbf{n}}(\mathcal{H}), \mathbf{n} \in \mathbb{N}_*^d\}$ , and a complete orthonormal system of eigenvectors  $\{\varphi_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}_*^d\}$  of  $L_E^2(\mathbb{R}^d, G_\alpha)$ , as follows:

$$(6.1) \quad H_1(\mathbf{x} - \mathbf{y}) = H(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H}) \varphi_{\mathbf{n}}(\mathbf{x}) \overline{\varphi_{\mathbf{n}}(\mathbf{y})},$$

where convergence holds in the  $L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)$  sense, i.e.,

$$(6.2) \quad \left\| H(\mathbf{x}, \mathbf{y}) - \sum_{\mathbf{n} \in \mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H}) \varphi_{\mathbf{n}} \otimes \overline{\varphi_{\mathbf{n}}} \right\|_{L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)}^2 = 0.$$

We can establish the following isometry  $\widehat{\mathcal{I}}_2$  between the Hilbert space  $L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)$ , and the two-Wiener chaos of the isonormal process  $X$  on  $H = L_E^2(\mathbb{R}^d, G_\alpha)$ , given by

$$(6.3) \quad X : h \in L_E^2(\mathbb{R}^d, G_\alpha) \rightarrow X(h) = \int_{\mathbb{R}^d}' h(\mathbf{x}) \frac{Z(d\mathbf{x})}{\|\mathbf{x}\|^{(d-\alpha)/2}}$$

(see [33], Chapter 9), considering the following identification between orthonormal bases of both spaces: For a given orthonormal basis  $\{\varphi_{\mathbf{n}} \otimes \overline{\varphi_{\mathbf{n}}}, \mathbf{n} \in \mathbb{N}_*^d\}$  of  $L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)$ , its image by such an isometry  $\widehat{\mathcal{I}}_2$  is defined as

$$(6.4) \quad \widehat{\mathcal{I}}_2(\varphi_{\mathbf{n}} \otimes \overline{\varphi_{\mathbf{n}}}) = \int_{\mathbb{R}^{2d}}'' [\varphi_{\mathbf{n}}(\mathbf{x}_1) \overline{\varphi_{\mathbf{n}}(\mathbf{x}_2)}] \frac{Z(d\mathbf{x}_1)}{\|\mathbf{x}_1\|^{(d-\alpha)/2}} \frac{Z(d\mathbf{x}_2)}{\|\mathbf{x}_2\|^{(d-\alpha)/2}},$$

which also defines an orthonormal basis in the two-Wiener chaos of the isonormal process  $X$  in (6.3).

From equations (6.1)–(6.2), by the orthonormality of the eigenvector basis  $\varphi_n \otimes \overline{\varphi}_n$  of  $L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)$ ,

$$\begin{aligned} \langle H, \varphi_{\mathbf{k}} \otimes \overline{\varphi}_{\mathbf{k}} \rangle_{L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)} &= \left\langle \sum_{\mathbf{n} \in \mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H}) \varphi_{\mathbf{n}} \otimes \overline{\varphi}_{\mathbf{n}}, \varphi_{\mathbf{k}} \otimes \overline{\varphi}_{\mathbf{k}} \right\rangle_{L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)} \\ (6.5) \quad &= \mu_{\mathbf{k}}(\mathcal{H}), \quad \forall \mathbf{k} \in \mathbb{N}_*^d. \end{aligned}$$

Again, from equations (6.1)–(6.2), and (6.5), considering the isometry  $\widehat{\mathcal{I}}_2$  in (6.4)

$$\begin{aligned} &\int_{\mathbb{R}^{2d}}'' H(\mathbf{x}_1, \mathbf{x}_2) \frac{Z(d\mathbf{x}_1)}{\|\mathbf{x}_1\|^{(d-\alpha)/2}} \frac{Z(d\mathbf{x}_2)}{\|\mathbf{x}_2\|^{(d-\alpha)/2}} \\ &= \sum_{\mathbf{n} \in \mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H}) \int_{\mathbb{R}^{2d}}'' \left[ \varphi_{\mathbf{n}}(\mathbf{x}_1) \overline{\varphi}_{\mathbf{n}}(\mathbf{x}_2) \right] \frac{Z(d\mathbf{x}_1)}{\|\mathbf{x}_1\|^{(d-\alpha)/2}} \frac{Z(d\mathbf{x}_2)}{\|\mathbf{x}_2\|^{(d-\alpha)/2}} \\ (6.6) \quad &= \sum_{\mathbf{n} \in \mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H}) H_2 \left( \int_{\mathbb{R}^d} \varphi_{\mathbf{n}}(\mathbf{x}) \frac{Z(d\mathbf{x})}{\|\mathbf{x}\|^{(d-\alpha)/2}} \right), \end{aligned}$$

where  $H_2$  denotes, as before, the second Hermite polynomial. Note that summation and integration can be swapped, in view of the convergence of the series (6.1) in the space  $L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)$ , and the referred isometry between  $L_E^2(\mathbb{R}^d, G_\alpha) \otimes L_E^2(\mathbb{R}^d, G_\alpha)$  and the two-Wiener chaos of isonormal process  $X$  introduced in (6.3) (see also equations (6.2)–(6.5)).

Note that

$$\int_{\mathbb{R}^{2d}}'' \varphi_{\mathbf{n}}(\mathbf{x}) \frac{Z(d\mathbf{x})}{\|\mathbf{x}\|^{(d-\alpha)/2}}, \quad \mathbf{n} \in \mathbb{N}_*^d,$$

are independent zero-mean Gaussian random variables with variance  $\int_{\mathbb{R}^{2d}} |\varphi_{\mathbf{n}}(\mathbf{x})|^2 G_\alpha(d\mathbf{x})$ , due to the orthogonality of the functions  $\varphi_{\mathbf{n}}$ ,  $\mathbf{n} \in \mathbb{N}_*^d$ , in the space  $L_E^2(\mathbb{R}^d, G_\alpha)$ . From equations (4.6) and (6.6),

$$S_\infty =_d c(d, \alpha) |D| \sum_{\mathbf{n} \in \mathbb{N}_*} \mu_{\mathbf{n}}(\mathcal{H}) (\varepsilon_{\mathbf{n}}^2 - 1).$$

Equation (4.15) is then obtained by setting  $\lambda_{\mathbf{n}}(S_\infty) = c(d, \alpha) |D| \mu_{\mathbf{n}}(\mathcal{H})$ .

**Proof of Proposition 1.** Let us consider  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  the Fourier and inverse Fourier transforms respectively defined on  $L^1(\mathbb{R}^d)$  and  $L^2(\mathbb{R}^d)$ . Consider an eigenpair  $(\mu, h)$  of the operator  $\mathcal{A}_\alpha$ , we have that  $\int_{\mathbb{R}^d} |h(\mathbf{y})|^2 \frac{1}{\|\mathbf{y}\|^{d-\alpha}} < \infty$ . Applying the inverse Fourier transform  $\mathcal{F}$  to both sides of the identity

$$\mu h = \mathcal{A}_\alpha h,$$

we get

$$\mu \mathcal{F}^{-1}(h) = \mathcal{F}^{-1}(\mathcal{A}_\alpha h) = c(d, \alpha) \mathcal{F}^{-1}(H_1 * H_2),$$

where, as before,

$$H_1(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2),$$

with kernel  $H$  being defined in equation (4.7), and  $H_2(\mathbf{y}) = \|\mathbf{y}\|^{-d+\alpha} h(\mathbf{y})$ . In the computation of this inverse Fourier transform, we note that  $H_1 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ . In order to apply the convolution theorem, we first perform the following decomposition:

$$H_2(\mathbf{y}) = \|\mathbf{y}\|^{-d+\alpha} h(\mathbf{y}) \mathbf{1}_{\mathcal{B}_1(\mathbf{0})}(\mathbf{y}) + \|\mathbf{y}\|^{-d+\alpha} h(\mathbf{y}) \mathbf{1}_{\mathbb{R}^d \setminus \mathcal{B}_1(\mathbf{0})}(\mathbf{y}) := H_2^-(\mathbf{y}) + H_2^+(\mathbf{y}),$$

where  $\mathcal{B}_1(\mathbf{0})$  denotes, as before, the ball with center zero and radius one in  $\mathbb{R}^d$ . Since

$$\int_{\mathbb{R}^d} h^2(\mathbf{y}) \|\mathbf{y}\|^{-d+\alpha} d\mathbf{y} < \infty,$$

$H_2^- \in L^1(\mathbb{R}^d)$ , and  $H_2^+ \in L^2(\mathbb{R}^d)$ . Applying the linearity of the convolution and Fourier transform, the convolution theorem for both  $L^1$  and  $L^2$  functions (see [45], and [40]) leads to

$$\begin{aligned} \mu \mathcal{F}^{-1}(h) &= c(d, \alpha) \mathcal{F}^{-1}(H_1 * H_2) = c(d, \alpha) [\mathcal{F}^{-1}(H_1 * H_2^-) + \mathcal{F}^{-1}(H_1 * H_2^+)] \\ &= c(d, \alpha) |D|^{-1} \mathbf{1}_D (\mathcal{F}^{-1}(H_2^- + H_2^+)) = c(d, \alpha) |D|^{-1} \mathbf{1}_D \mathcal{F}^{-1} H_2, \end{aligned} \quad (6.7)$$

where we have considered equations (4.5) and (4.7). From (6.7), we can see that the support of  $\mathcal{F}^{-1}(h)$  is contained in  $D$ , for any eigenfunction  $h$  of  $\mathcal{A}_\alpha$ . The convolution theorem for generalized functions (see [45]) can be applied again to  $H_2$ , since  $h$  has compact support. By (4.17),  $G_\alpha(d\mathbf{x}) = g_\alpha(\mathbf{x})d\mathbf{x}$ , with  $g_\alpha(\mathbf{x}) = \|\mathbf{x}\|^{-d+\alpha}$ . Then,

$$h(\mathbf{y}) \|\mathbf{y}\|^{-d+\alpha} = \mathcal{F}(\mathcal{F}^{-1}(h) * \mathcal{F}^{-1}(g_\alpha))(\mathbf{y}).$$

Therefore, in equation (6.7), we obtain

$$\begin{aligned} \mu \mathcal{F}^{-1}(h) &= c(d, \alpha) |D|^{-1} \mathbf{1}_D \mathcal{F}^{-1} [\mathcal{F}(\mathcal{F}^{-1}(h) * \mathcal{F}^{-1}(g_\alpha))] \\ &= c(d, \alpha) |D|^{-1} \mathbf{1}_D (\mathcal{F}^{-1}(h) * \mathcal{F}^{-1}(g_\alpha)). \end{aligned} \quad (6.8)$$

The inverse Fourier transform  $\mathcal{F}^{-1}$  of  $g_\alpha(\mathbf{y}) = \|\mathbf{y}\|^{-d+\alpha}$  is obtained from equation (3.4) in Lemma 2 (see also [41], p.117):

$$\mathcal{F}^{-1}(g_\alpha)(\mathbf{z}) = \frac{1}{c(d, \alpha) \|\mathbf{z}\|^\alpha} = \frac{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(\frac{d-\alpha}{2})} \|\mathbf{z}\|^{-\alpha}.$$

Applying (6.8) and this last relation, we finally obtain that, for an eigenpair  $(\mu, h)$  of  $\mathcal{A}_\alpha$ ,

$$\mu \mathcal{F}^{-1}(h)(\mathbf{z}) = |D|^{-1} \mathbf{1}_D(\mathbf{z}) \int_D \|\mathbf{z} - \mathbf{y}\|^{-\alpha} \mathcal{F}^{-1}(h)(\mathbf{y}) d\mathbf{y}, \quad (6.9)$$

since, as commented before,  $\mathcal{F}^{-1}(h)$  is supported on  $D$ . Thus, if  $(\mu, h)$  is an eigenpair of  $\mathcal{A}_\alpha$ , then  $(\mu, \mathcal{F}^{-1}(h))$  is an eigenpair for  $|D|^{-1} \mathcal{K}_\alpha$  on  $L^2(D)$ . The converse assertion also holds. Hence, there exists a one-to-one correspondence between eigenpairs of  $\mathcal{A}_\alpha$  and  $|D|^{-1} \mathcal{K}_\alpha$ , which preserves the eigenvalues, as we wanted to prove.

## APPENDIX B

The proof of Theorem 4 is based on Lemma 4.1 in [48], which is now formulated.

**Lemma 4.** *Define the function  $G_c(x) = \sum_{k=1}^{\infty} x^{c_k}$ , with  $c = \{c_n\}$  being a positive strictly increasing sequence such that  $c_n \sim \beta n^\alpha$ , as  $n \rightarrow \infty$ , for some  $1/2 < \alpha < 1$ , and constant  $\beta > 0$ . Then,*

$$\begin{aligned} G_c(x) &\sim x^{c_1}, \quad \text{as } x \rightarrow 0 \\ G_c(x) &\sim \frac{1}{\alpha \beta^{1/\alpha}} \Gamma\left(\frac{1}{\alpha}\right) (1-x)^{-1/\alpha}, \quad \text{as } x \rightarrow 1. \end{aligned} \quad (6.10)$$

## APPENDIX C

An alternative proof of the boundedness of the probability density of  $S_\infty$ , based on the series representation given in Corollary 1 is derived, and an upper bound is also provided.

**Proof of boundedness of the probability density of  $S_\infty$ .** From Corollary 2, there exist two indexes  $\mathbf{k}_0$  and  $\mathbf{k}_1$  such that  $\lambda_{\mathbf{k}_0}(S_\infty) > \lambda_{\mathbf{k}_1}(S_\infty)$ . Then,

$$S_\infty = \sum_{\mathbf{k} \in \mathbb{N}_*^d} \lambda_{\mathbf{k}}(S_\infty) (\varepsilon_{\mathbf{k}}^2 - 1) = \lambda_{\mathbf{k}_0}(S_\infty)(\varepsilon_{\mathbf{k}_0}^2 - 1) + \lambda_{\mathbf{k}_1}(S_\infty)(\varepsilon_{\mathbf{k}_1}^2 - 1) + \eta.$$

where

$$\eta = \sum_{\mathbf{k} \in \mathbb{N}_*^d, \mathbf{k} \neq \mathbf{k}_0, \mathbf{k}_1} \lambda_{\mathbf{k}}(S_\infty) (\varepsilon_{\mathbf{k}}^2 - 1).$$

Thus,

$$S_\infty = \lambda_{\mathbf{k}_1}(S_\infty)(\beta \varepsilon_{\mathbf{k}_0}^2 + \varepsilon_{\mathbf{k}_1}^2) - (\lambda_{\mathbf{k}_0}(S_\infty) + \lambda_{\mathbf{k}_1}(S_\infty)) + \eta,$$

where  $\beta = \lambda_{\mathbf{k}_0}(S_\infty)/\lambda_{\mathbf{k}_1}(S_\infty)$ .

The random variables  $\varepsilon_{\mathbf{k}_0}^2$  and  $\varepsilon_{\mathbf{k}_1}^2$  are independent. Since the density of  $\varepsilon_{\mathbf{k}_1}^2$  is of the form

$$f_{\varepsilon_{\mathbf{k}_1}^2}(x) = \frac{1}{\Gamma(\frac{1}{2})\sqrt{2}} x^{-1/2} e^{-x/2}, \quad x > 0,$$

and the density of  $\beta \varepsilon_{\mathbf{k}_0}^2$  is given by

$$f_{\beta \varepsilon_{\mathbf{k}_0}^2}(x) = \frac{1}{\beta \Gamma(\frac{1}{2})\sqrt{2}} (x/\beta)^{-1/2} e^{-x/2\beta}, \quad x > 0,$$

noting that  $\beta = \frac{\lambda_{\mathbf{k}_0}(S_\infty)}{\lambda_{\mathbf{k}_1}(S_\infty)} > 1$ , then the density of  $\varsigma = \beta \varepsilon_{\mathbf{k}_0}^2 + \varepsilon_{\mathbf{k}_1}^2$  satisfies

$$\begin{aligned} f_\varsigma(u) &= \int_0^u f_{\varepsilon_{\mathbf{k}_1}^2}(u-x) f_{\beta \varepsilon_{\mathbf{k}_0}^2}(x) dx \\ &= \frac{e^{-u/2}}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \int_0^u (u-x)^{-1/2} e^{\frac{x}{2}} x^{-1/2} e^{-\frac{x}{2\beta}} dx = \\ &\quad [1 - \frac{1}{\beta} > 0] \\ &= \frac{e^{-u/2}}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \int_0^u (u-x)^{-1/2} e^{\frac{x}{2}(1-\frac{1}{\beta})} x^{-1/2} dx \\ &\leq \frac{e^{-u/2} e^{\frac{u}{2}(1-\frac{1}{\beta})}}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \int_0^u (u-x)^{-1/2} x^{-1/2} dx \\ (6.11) \quad &\leq e^{-\frac{u}{2\beta}} \frac{B(\frac{1}{2}, \frac{1}{2})}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \leq \frac{1}{2\sqrt{\beta}} = \frac{1}{2\sqrt{\frac{\lambda_{\mathbf{k}_0}(S_\infty)}{\lambda_{\mathbf{k}_1}(S_\infty)}}} \leq \frac{1}{2}. \end{aligned}$$

As the convolution of a bounded density with other is bounded, we then obtain the desired result.

## REFERENCES

- [1] Adler, R.J., Taylor, J.E: Random Fields and Geometry. Springer Monographs in Mathematics. Springer, New York (2007)
- [2] Albin, J.M.P: A note on Roseblatt distributions. Statist. Probab. Letters 40, 83–91 (1998)
- [3] Anh, V.V., Leonenko, N.N., Olenko, A: On the rate of convergence to Rosenblatt-type distribution. J. Math. Anal. Appl. 425, 111–132 (2015)
- [4] Aoyama, T., Maejima, M., Veda, Y: Several forms of stochastic integral representations of gamma random variables and related topics. Prob. and Math. Statist. 31, 99–118 (2011)
- [5] Applebaum, D: Lévy Processes and Stochastic Calculus. Cambridge University Press, Cambridge, UK (2004)
- [6] Arendt, W., Schleich, W: Mathematical Analysis of Evolution, Information, and Complexity. Wiley, New York (2009)

- [7] Barndorff-Nielsen, O.E., Maejima, M., Sato, K: Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. *Bernoulli* 12, 1–33 (2006)
- [8] Bondesson, L: Generalized Gamma Convolutions and Related Classes of Distributions and Densities. *Lecture Notes in Statistics* 76. Springer, Berlin (1992)
- [9] Caetano, A.M: Approximation by functions of compact support in Besov-Triebel-Lizorkin spaces on irregular domains. *Studia Mathematica* 142, 47–63 (2000)
- [10] Chen, Z.Q. and Song, R. Two-sided eigenvalue estimates for subordinate processes in domains. *Journal of Functional Analysis* 226, 90–113 (2005)
- [11] Da Prato, G., Zabczyk, J: *Second Order Partial Differential Equations in Hilbert Spaces*. Cambridge University Press, Cambridge (2002)
- [12] Dobrushin, R.L., Major, P: Non-central limit theorem for non-linear functionals of Gaussian fields. *Z. Wahrsch. Verw. Geb.* 50, 1–28 (1979)
- [13] Dostanic, M.R: Spectral properties of the operator of Riesz potential type. *Proceedings of the American Mathematical Society* 126, 2291–2297 (1998)
- [14] Doukhan, P., León J.R., Soulier P: Central and non-central limit theorems for quadratic forms of a strongly dependent Gaussian field. *Braz J. Probab. Stat.* 10, 205–223 (1996)
- [15] Fox, R., Taqqu, M.S: Noncentral limit theorems for quadratic forms in random variables having long-range dependence. *Ann. Probab.* 13, 428–446 (1985)
- [16] Gordon, C. L., Webb, D.L., Wolpert, S: One cannot hear the shape of a drum. *Bull. Amer. Math. Soc.* 27, 134–138 (1992)
- [17] Ivanov, A.V., Leonenko, N.N: *Statistical Analysis of Random Fields*. Kluwer Academic Publishers, Dordrecht (1989)
- [18] James, L.F., Roynette, B., Yor, M: Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Prob. Surveys* 5, 346–415 (2008)
- [19] Jurek, Z.J., Vervaat, W: An integral representation for selfdecomposable Banach space valued random variable. *Z. Wahrsch. Verw. Gebiete* 62, 247–262 (1983)
- [20] Lapidus M.L: Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture. *Trans. Amer. Math. Soc.* 325, 465–529 (1991)
- [21] Leonenko, N., Olenko, A.: Tauberian and Abelian theorems for long-range dependent random fields. *Methodology and Computing in Applied Probability* 15, 715–742 (2013)
- [22] Leonenko, N., Olenko, A: Sojourn measures of Student and Fisher-Snedecor random fields. *Bernoulli* 20, 1454–1483 (2014)
- [23] Leonenko, N., Ruiz-Medina, M.D. and Taqqu, M.S. Non-Central limit theorems for random fields subordinated to gamma-correlated random fields. *Bernoulli* (2016) (to appear <http://www.bernoulli-society.org/index.php/publications/bernoulli-journal/bernoulli-journal-papers>)
- [24] Leonenko, N.N., Taufer, E: Weak convergence of functionals of stationary long memory processes to Rosenblatt-type distributions. *Journal of Statistical Planning and Inference* 136, 1220–1236 (2006)
- [25] Lim, S.C., Teo, L.P: Analytic and asymptotic properties of multivariate generalized Linnik’s probability density. *J. Fourier Anal. Appl.* 16, 715–747 (2010)
- [26] Lukacs, E: *Characteristic Functions* (Second ed.). Griffin, London (1970)
- [27] Maejima, M., Tudor, C.A: On the distribution of the Rosenblatt process. *Statistics and Probability Letters* 83, 1490–1495 (2013)
- [28] Major, P: *Multiple Wiener-Ito Integrals: With Applications to Limit Theorems*. *Lecture Notes in Mathematics*. Springer, New York (1981)
- [29] Mathai, A.M., Provost, S.B.: *Quadratic Forms in Random Variables*. Dekker, New York (1992)
- [30] Müller, W: Relative zeta functions, relative determinants and scattering theory. *Commun. Math. Phys.* 192, 309–347 (1998)
- [31] Park, J., Wojciechowski, K.P: Adiabatic decomposition of the  $\zeta$ -determinant of the Dirac Laplacian I. The case of invertible tangential operator. With an appendix by Y. Lee. *Comm. Partial Differential Equations* 27, 1407–1435 (2002a)
- [32] Park, J., Wojciechowski, K.P: Scattering theory and adiabatic decomposition of the  $\zeta$ -determinant of the Dirac Laplacian. *Math. Res. Lett.* 9, 17–25 (2002b)
- [33] Peccati, G., Taqqu, M.S: *Wiener Chaos: Moments, Cumulants and Diagrams*. Springer, New York (2011)
- [34] Reed, M., Simon, B: *Methods of Modern Mathematical Physics I: Functional analysis*. Academic Press. New York (1980)
- [35] Rosenblatt, M: Independence and dependence. *Proc. 4th Berkeley Symp. Math. Stat. Probab. Univ. Calif. Press*, pp. 411–433 (1961)
- [36] Rosenblatt, M: Some limit theorems for partial sums of quadratic forms in stationary Gaussian variables. *Z. Wahrsch. verw. Gebiete* 49, 125–132 (1979)

- [37] Sato, K.I: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)
- [38] Seneta E: Regularly Varying Functions. Springer, Berlin (1976)
- [39] Simon, B: Trace ideals and their applications. Mathematical Surveys and Monographs 120. Providence, RI: American Mathematical Society (AMS) (2005)
- [40] Stade, E: Fourier Analysis. Pure and Applied Mathematics. Wiley-Interscience, John Wiley & Sons, New York (2005)
- [41] Stein, E.M: Singular Integrals and Differential Properties of Functions. Princenton, University Press, New Jersey (1970)
- [42] Taqqu, M.S: Weak-convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrsch. Verw. Gebiete 31, 287–302 (1975)
- [43] Taqqu, M.S: Convergence of integrated processes of arbitrary Hermite rank. Z. Wahrsch. Verw. Gebiete 50, 53–83 (1979)
- [44] Thorin, O: An extension of the notion of a generalized  $\Gamma$ -convolution. Scand. Actuarial J., 141—149 (1978)
- [45] Triebel, H: Interpolation Theory, Function Spaces, Differential Operators. North-Holland, New York (1978)
- [46] Triebel, H: Fractals and Spectra. Birkhäuser, New York (1997)
- [47] Triebel, H., Yang, D: Spectral theory of Riesz potentials on quasi-metric spaces. Math. Nachr. 238, 160–184 (2001)
- [48] Veillette, M.S., Taqqu, M.S: Properties and numerical evaluation of Rosenblatt distribution. Bernoulli 19, 982–1005 (2013)
- [49] Widom, H: Asymptotic behavior of the eigenvalues of certain integral equations. Trans. Amer. Math. Soc. 109, 278–295 (1963)
- [50] Yamazato, M: Unimodality of infinitely divisible distribution functions of class  $\mathcal{L}$ . Ann. Prob. 6, 523–531 (1978)
- [51] Zähle, M: Riesz potentials and Liouville operators on fractals. Potential Analysis 21, 193–208 (2004)

(N.N. Leonenko) CARDIFF SCHOOL OF MATHEMATICS, SENGHENNYDD ROAD, CARDIFF CF24 4AG, UNITED KINGDOM

*E-mail address:* `LeonenkoN@cardiff.ac.uk`

(M.D. Ruiz-Medina) UNIVERSITY OF GRANADA, DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH, CAMPUS FUENTE NUEVA S/N, E-18071 GRANADA, SPAIN

*E-mail address:* `mrui@ugr.es`

(M.S. Taqqu) DEPARTMENT OF MATHEMATICS AND STATISTICS, 111 CUMMINGTON ST., BOSTON UNIVERSITY, BOSTON, MA 02215, USA

*E-mail address:* `murad@bu.edu`